HECKE ORBITS ON SHIMURA VARIETIES OF HODGE TYPE

MARCO D'ADDEZIO AND POL VAN HOFTEN

Abstract. We prove the Hecke orbit conjecture of Chai–Oort for Shimura varieties of Hodge type at primes of good reduction, under a mild assumption on the size of the prime. Our proof uses a non-abelian generalisation of Serre–Tate coordinates for deformation spaces of central leaves in such Shimura varieties, constructed using work of Caraiani–Scholze and Kim. We use these coordinates to give a new interpretation of Chai–Oort's notion of strongly Tate-linear subspaces of these deformation spaces. This lets us prove upper bounds on the local monodromy of these subspaces using the Cartier–Witt stacks of Bhatt–Lurie. We also prove a rigidity result in the style of Chai–Oort for strongly Tate-linear subspaces. Another main ingredient of our proof is a new result on the local monodromy groups of F-isocrystals "coming from geometry" which refines Crew's parabolicity conjecture.

Contents

1. Introduction

1.1. **The Hecke orbit conjecture.** Let p be a prime number and g a positive integer. Problem 15 on Oort's 1995 list of open problems in algebraic geometry, [65], is the following conjecture.

Conjecture 1. *Let $x = (A_x, \lambda)$ be an $\overline{\mathbb{F}}_p$-point of the moduli space $\mathcal{A}_g$ of principally polarised abelian varieties of dimension g over $\overline{\mathbb{F}}_p$. The Hecke orbit of x, consisting of all points $y \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ corresponding to principally polarised abelian varieties related to (A_x, λ) by symplectic isogenies, is Zariski dense in the Newton stratum of $\mathcal{A}_g$ containing x.*

Date: September 1, 2023.
2010 *Mathematics Subject Classification.* Primary 11G18; Secondary 14G35.

More generally, for the special fibre of a Shimura variety of Hodge type at a prime of good reduction, one expects that the isogeny classes are Zariski dense in the Newton strata containing them. This article contains a proof of this expectation under the assumption that p is not too small with respect to the given Shimura datum; for $\mathcal{A}_g$ this comes down to the assumption that $p \geq 2g$.

1.1.1. There is a refined version of Conjecture 1, also due to Oort, which considers instead the prime-to-p Hecke-orbit of x, consisting of all $y \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ related to x by prime-to-p symplectic isogenies. In this case, the quasi-polarised p-divisible group $(A_x[p^\infty], \lambda)$ is constant on prime-to-p Hecke orbits (not just constant up to isogeny). Therefore, the prime-to-p Hecke orbit of x is contained in the central leaf

$$C(x) = \left\{ y \in \mathcal{A}_g(\overline{\mathbb{F}}_p) \mid A_y[p^\infty] \simeq_\lambda A_x[p^\infty] \right\},$$

where $\simeq_\lambda$ denotes a symplectic isomorphism. Oort proved in [64] that $C(x)$ is a smooth closed subvariety of the Newton stratum of $\mathcal{A}_g$ containing x. He also conjectured that the prime-to-p Hecke orbit of x was Zariski dense in the central leaf $C(x)$. This conjecture is known as the *Hecke orbit conjecture* (for $\mathcal{A}_g$). Thanks to the Mantovan–Oort product formula, [56, 64], the Hecke orbit conjecture implies Conjecture 1.

Central leaves and prime-to-p Hecke orbits can also be defined for the special fibres of Shimura varieties of Hodge type at primes of good reduction, by work of Hamacher and Kim [33, 47]. The Hecke orbit conjecture for Shimura varieties of Hodge type then predicts that the prime-to-p Hecke orbits of points are Zariski dense in the central leaves containing them (see question 8.2.1 of [55] and Conjecture 3.2 of [13]).

The Hecke orbit conjecture naturally splits up into a discrete part and a continuous part. The discrete part states that the prime-to-p Hecke orbit of x intersects each connected component of $C(x)$, whereas the continuous part states that the Zariski closure of the prime-to-p Hecke orbit of x is equidimensional of the same dimension as $C(x)$. The discrete part of the conjecture is Theorem C of [55] (see [40] for related results). In this paper, we will focus instead on the continuous part of the conjecture.

1.2. **Main result.** Let (G, X) be a Shimura datum of Hodge type with reflex field E, and assume for simplicity that G^{ad} is $\mathbb{Q}$-simple throughout this introduction. Let p be a prime such that $G_{\mathbb{Q}_p}$ is quasi-split and split over an unramified extension, let $U_p \subseteq G(\mathbb{Q}_p)$ be a hyperspecial subgroup, and let $U^p \subseteq G(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup. We choose a place v of E dividing p and we write $\mathrm{Sh}_{G,U}$ for the geometric special fibre of the canonical integral model over $\mathcal{O}_{E,v}$ of the Shimura variety $\mathbf{Sh}_{G,U}$ over E of level $U := U^p U_p$. The canonical integral model exists by [52] and [49] for $p = 2$. Let $C \subseteq \mathrm{Sh}_{G,U}$ be a central leaf as constructed in [33] (cf. [47]) and let $h(G)$ be the Coxeter number of G.

Theorem I (Theorem 8.3.2)**.** *Let $Z \subseteq C$ be a non-empty reduced closed subvariety that is stable under the prime-to-p Hecke operators. If $p \geq h(G)$, then $Z = C$.*

When $\mathrm{Sh}_{G,U}$ is a Siegel modular variety, this result (without any conditions on p) is due to Chai–Oort, see their forthcoming book [19] for the continuous part and [17] for the discrete part. Their proofs do not generalise to more general Shimura varieties because they rely on the existence of hypersymmetric points in Newton strata, which is usually false for Shimura varieties of Hodge type, see [30]. Moreover, their proof of the continuous part of the conjecture relies on the fact that any point $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ is contained in a large Hilbert modular variety, and they use work of Yu–Chai–Oort, [77], on the Hecke orbit conjecture for Hilbert modular varieties at (possibly ramified) primes. There are many other partial results, e.g. for prime-to-p Hecke orbits of hypersymmetric points in the PEL case, [75], or for prime-to-p Hecke orbits of (μ)-ordinary points, [10, 39, 59, 69, 78].

We also prove that isogeny classes are dense in the Newton strata containing them, see Theorem 8.4.1. Moreover, we prove results about ℓ-adic Hecke orbits for primes $\ell \neq p$ generalising work of Chai, [12], in the Siegel case, see Theorem 8.6.1.

Remark 1.2.1. The condition that $p \geq h(G)$, as well as the assumption that G^{ad} is $\mathbb{Q}$-simple, can be relaxed at the expense of introducing more notation, see Theorem 8.3.2 for a precise statement. For $\mathcal{A}_g$, it for example suffices to assume that $p \geq$ the number of distinct slopes of the Newton polygon corresponding to the Newton stratum that the central leaf lies in.

Remark 1.2.2. Theorem 8.10 of [8], which gives a potentially good reduction criterion for K3 surfaces, is conditional on the Hecke orbit conjecture for certain orthogonal Shimura varieties, see Conjecture 8.2 of [*ibid.*]. In Section 8.5 we explain that our results can be used to prove this conjecture under mild assumptions on p.

1.3. **Local monodromy of F-isocrystals.** One of the main tools of the proof is the theory of *monodromy groups of F-isocrystals*, defined in [21]. We will prove a new result about the local monodromy groups of F-isocrystals "coming from geometry", which should be of independent interest. This result is used to prove that the local monodromy groups of the crystalline Dieudonné modules of the universal p-divisible groups over $Z \subseteq C$ (notation as in Theorem I) are "big". We explain it in a more general setting.

Let X be a smooth irreducible variety over a perfect field with a rational point x (for simplicity) and let $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}^\dagger})$ be a semi-simple overconvergent F-isocrystal over X with constant Newton polygon. Since the Newton polygon is constant, the associated F-isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ admits the slope filtration. The main result of [23] tells us that the *monodromy group* of $\mathcal{M}$ (Definition 2.3.4), denoted by $G(\mathcal{M}, x)$, is the parabolic subgroup $P \subseteq G(\mathcal{M}^\dagger, x)$ associated to the slope filtration. We refine that result for the local monodromy group at x. Let $X^{/x}$ be the formal completion of X at x and write $G(\mathcal{M}^{/x}, x)$ for the monodromy group of the restriction of $\mathcal{M}$ to $X^{/x}$ (see Notation 2.3.5).

Theorem II. *The monodromy group $G(\mathcal{M}^{/x}, x)$ of the restriction of $\mathcal{M}$ to $X^{/x}$ is the unipotent radical of the monodromy group $G(\mathcal{M}, x)$.*

When $\mathcal{M}$ is the crystalline Dieudonné-module of an ordinary p-divisible group, this result is proved by Chai in [11] by doing explicit computations with Serre–Tate coordinates. Our proof builds instead on the techniques developed in [23] and uses new descent results for isocrystals from [26, 57]. Since $X^{/x}$ is geometrically simply connected, each isocrystal underlying an isoclinic F-isocrystal over $X^{/x}$ is trivial by Theorem 6 of [2]. This already implies that $G(\mathcal{M}^{/x}, x)$ is unipotent. To relate $G(\mathcal{M}^{/x}, x)$ and the unipotent radical of $G(\mathcal{M}, x)$, we pass through the respective generic points.

Write k for the function field of X and k_x for the function field of $X^{/x}$. We first prove in Theorem 3.2.5 that if we pass from X to $\operatorname{Spec} k$ we do not change the monodromy group of $\mathcal{M}$, as in the étale setting. Then we show that if we extend k to k^{sep} the monodromy group of $\mathcal{M}$ becomes the unipotent radical of $G(\mathcal{M}, x)$ (Proposition 3.4.2). This means that the extension of scalars from k to k^{sep} kills precisely a Levi subgroup of $G(\mathcal{M}, x)$. Subsequently, using the fact that the field extension $k \subseteq k_x$ is formally étale we show that when we extend F-isocrystals from k^{sep} to k_x^{sep} their slope filtration does not acquire new splittings (Proposition 3.3.5). This is enough to prove that the local monodromy group $G(\mathcal{M}^{/x}, x)$ is the same as the monodromy group over k^{sep}. By the previous part of the argument we then deduce that $G(\mathcal{M}^{/x}, x)$ is precisely the unipotent radical of $G(\mathcal{M}, x)$.

1.4. **An overview of the proof of Theorem I.** The overall structure of the proof of Theorem I is similar to the proof of the ordinary Hecke orbit conjecture in [39] and is based on a strategy implicit in the work of Chai–Oort and sketched to us by Chai in a letter.

To explain the proof we first need to establish some notation. Let $Z \subseteq C \subseteq \mathrm{Sh}_{G,U,[b]}$ be a reduced closed subvariety that is stable under the prime-to-p Hecke operators as in the statement of Theorem I. The Newton stratum $\mathrm{Sh}_{G,U,[b]} \subseteq \mathrm{Sh}_{G,U}$ corresponds to an element $[b] \in B(G_{\mathbb{Q}_p})$ which has an associated Newton (fractional) cocharacter ν_b. Attached to this cocharacter is a parabolic subgroup P_{ν_b} with unipotent radical U_{ν_b}.

Let Z^{sm} be the smooth locus of Z. Then Corollary 3.3.3 of [39] tells us that the monodromy group of the crystalline Dieudonné module $\mathcal{M}$ of the universal p-divisible group over Z^{sm} is isomorphic to P_{ν_b}. Theorem II then tells us that for $x \in Z^{\mathrm{sm}}(\overline{\mathbb{F}}_p)$, the monodromy of $\mathcal{M}$ over $Z^{/x} := \mathrm{Spf}\, \widehat{\mathcal{O}}_{Z,x}$ is equal to U_{ν_b}. We are going to leverage this fact to show that $Z^{/x} = C^{/x}$, which will allow us to conclude that Z^{sm} and hence Z is equidimensional of the same dimension as C.

For this we will construct generalised Serre–Tate coordinates on the formal completion $C^{/x} := \mathrm{Spf}\, \widehat{\mathcal{O}}_{C,x}$. To be precise, we will show that there is a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra[1] $\mathfrak{a}^+$ over $\breve{\mathbb{Z}}_p$ governing the structure of $C^{/x}$. For example, if C is the ordinary locus in $\mathcal{A}_g$ then $C^{/x}$ is a p-divisible formal group by the classical theory of Serre–Tate coordinates, and $\mathfrak{a}^+ = \mathbb{D}(C^{/x})$ is its Dieudonné module, equipped with the trivial Lie bracket.

More generally the perfection of $C^{/x}$ admits the structure of a (functor to) nilpotent Lie $\mathbb{Q}_p$-algebra(s) whose Dieudonné module is $\mathfrak{a} := \mathfrak{a}^+[\frac{1}{p}]$. More canonically, the perfection of $C^{/x}$ is a trivial torsor for a unipotent formal group $\tilde{\Pi}(\mathfrak{a})$ related to the aforementioned nilpotent Lie algebra by the Baker–Campbell–Hausdorff formula. In the Siegel case, this unipotent formal group is the identity component of the group of self-quasi isogenies (compatible with the polarisation up to a scalar) of the p-divisible group $A_x[p^\infty]$. These results come from the perspective of Caraiani–Scholze, [9], on C and the perspective of Kim, [47], on $C^{/x}$.

We note that our generalisation of Serre–Tate coordinates is different from the notion of a p-divisible cascade given by Moonen in [62]. Our alternative definition is necessary in our work since in the Hodge type case the deformation spaces we consider generally do not have a cascade structure. Already in the μ-ordinary case, all we can hope for is a *shifted subcascade* in the sense of [70] (cf. [42]) and, as far as we can see, there is no way to run our arguments with shifted subcascades.

The Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra $\mathfrak{a}$ turns out to be isomorphic to $\mathrm{Lie}\, U_{\nu_b}$ equipped with a natural F-structure coming from $[b]$. If $\mathfrak{b} \subseteq \mathfrak{a} = \mathrm{Lie}\, U_{\nu_b}$ is an F-stable Lie subalgebra, we construct a formally smooth closed formal subscheme $Z(\mathfrak{b}^+) \subseteq C^{/x}$ (Definition 4.2.14), which is strongly Tate-linear in the sense of Chai–Oort, [15]. The construction is such that when $\mathfrak{b} = \mathfrak{a}$, the formal subscheme $Z(\mathfrak{b}^+)$ is $C^{/x}$ itself. It turns out that $Z^{/x}$ admits such a description.

Theorem 1.4.1 (Theorem 7.1.1)**.** *There is an F-stable Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{a}$ such that $Z^{/x} = Z(\mathfrak{b}^+)$.*

To prove Theorem I, we are then reduced to show that the formal subschemes $Z(\mathfrak{b}^+)$ for $\mathfrak{b} \subsetneq \mathfrak{a}$ have small monodromy. In this direction, we prove the following result:

Theorem 1.4.2 (Theorem 6.1.1)**.** *The Lie algebra of the monodromy group of $\mathcal{M}$ over $Z(\mathfrak{b}^+)$ is contained in $\mathfrak{b}$.*

[1] See Definition 4.2.1.

By the previous discussion, we know that the Lie algebra of the monodromy group of $\mathcal{M}$ over $Z^{/x}$ is equal to $\mathfrak{a}$. Therefore, if $Z^{/x} = Z(\mathfrak{b}^+)$ for some $\mathfrak{b}$, then $\mathfrak{a} \subseteq \mathfrak{b} \subseteq \mathfrak{a}$, so that $Z^{/x} = Z(\mathfrak{b}^+) = Z(\mathfrak{a}^+) = C^{/x}$.

The proof of Theorem 1.4.2 uses the Cartier–Witt stacks of Bhatt–Lurie [3] in combination with the interpretation of $C^{/x}$ as a formal deformation space of the trivial torsor for a certain group scheme $\Pi(\mathfrak{a}^+)$, due to Chai–Oort [15] in the PEL case. In particular, we show that the closed formal subscheme $Z(\mathfrak{b}^+) \subseteq C^{/x}$ can be identified with the formal deformation space of the trivial torsor for a certain closed subgroup $\Pi(\mathfrak{b}^+) \subseteq \Pi(\mathfrak{a}^+)$. Our argument with Cartier–Witt stacks happens in Section 6.

Theorem 1.4.1 is related to rigidity results for p-divisible formal groups of Chai, [14], and to rigidity results for biextensions of p-divisible formal groups of Chai–Oort, [18]. We would also like to mention unpublished work of Tao Song which proves rigidity results in the case of p-divisible 4-cascades.

The proof of Theorem 1.4.1, which happens in Section 7, is heavily inspired by the proof of the rigidity result for biextensions of [18] and uses some of the results of [*ibid.*] on the topological commutative algebra of completions of perfections of power series rings over $\overline{\mathbb{F}}_p$, which are summarised in Appendix A. However, it is not entirely straightforward to translate the proofs of [18], which happen with biextensions, to the setting of unipotent groups and Dieudonné–Lie algebras.

Remark 1.4.3. The unipotent formal group $\tilde{\Pi}(\mathfrak{a})$ for $\mathfrak{a} = \operatorname{Lie} U_{\nu_b}$ is closely related to the "unipotent group diamond" $\tilde{G}_b^{>0}$ introduced in Chapter III.5 of Fargues–Scholze, [29]. To be precise, there should be an isomorphism $\tilde{\Pi}(\mathfrak{a})^{\diamondsuit} \simeq \tilde{G}_b^{>0}$ of v-sheaves in groups over $\operatorname{Spd} \overline{\mathbb{F}}_p$.

1.5. **Structure of the article.** In Section 2 we cover some background theory on p-divisible groups, isocrystals, and prove some technical results about formal schemes. We prove Theorem II in Section 3. In Section 4 we discuss internal hom p-divisible groups and groups of (quasi-)automorphisms of p-divisible groups, and introduce Dieudonné–Lie algebras. In Section 5 we discuss central leaves for Shimura varieties of Hodge type and describe their deformation theory in terms of Dieudonné–Lie algebras using work of Kim, [47]. We also relate our perspective on the deformation theory of central leaves to the one of Chai–Oort, [15]. We prove Theorem 1.4.2 in Section 6 and Section 7 is devoted entirely to the proof of Theorem 1.4.1. Additionally, every section except for Section 2 has its own introduction which describes in more detail what happens in that section.

1.6. **Acknowledgements.** We are indebted to Ching-Li Chai and Frans Oort for the ideas they developed and we used for our work on the Hecke orbit conjecture. The first named author would also like to thank Ching-Li Chai for his letter on the relation between the parabolicity conjecture and the Hecke orbit conjecture. We are grateful to Brian Conrad, Sean Cotner, Andrew Graham, and Richard Taylor for many helpful discussions about this work. We also thank Hélène Esnault and Matteo Tamiozzo for their comments on an earlier version of this article.

The first named author was funded by the Deutsche Forschungsgemeinschaft (DA-2534/1-1, project ID: 461915680).

2. Preliminaries

In this section we will introduce some notation and recall some definitions about Dieudonné-modules and isocrystals. First, we prove some preliminary technical result about formal schemes that we will use in Section 7.

2.1. **Formal schemes over a field.** In this text we will make use of the theory of formal algebraic spaces as developed in Chapter 86 of [73]. The admissible topological rings we will consider will be *adic rings* with a finitely generated ideal of definition.

Definition 2.1.1. A *pre-adic* ring* is a topological ring R endowed with the I-adic topology for some finitely generated ideal I. An *adic* ring* is an I-adically complete pre-adic* ring.

If R is an adic* ring over a field κ (not asked to be perfect) with finitely generated ideal of definition I, we write $h_{\mathrm{Spf}\,R} : \mathrm{Alg}_\kappa^{\mathrm{op}} \to \mathrm{Sets}$ for the associated functor of points, where $\mathrm{Alg}_\kappa^{\mathrm{op}}$ is the opposite of the category of κ-algebras endowed with the fpqc topology. This functor of points is defined as

$$\varinjlim_n h_{\mathrm{Spec}\,R/I^n}.$$

If S is another adic* ring, by [73, Lemma 0AN0], a homomorphism of functors $f : h_{\mathrm{Spf}\,S} \to h_{\mathrm{Spf}\,R}$ comes uniquely from a continuous homomorphism $f : R \to S$. From now on, we will often implicitly identify $\mathrm{Spf}\,R$ and $h_{\mathrm{Spf}\,R}$.

2.1.2. *Scheme-theoretic images.* Let $f : R \to S$ be a morphism of adic* rings with kernel K (a closed ideal of R by the completeness of S). If I is a finitely generated ideal of definition of R, then we can form the ring

$$T := \varprojlim_n R/(I^n + K),$$

which is just the J-adic completion of R/K where $J := I/K$. The ring T is J-adically complete because J is finitely generated. By Lemma 0APT, the morphism $R \to T$ is surjective. This implies that $T = R/K$, or in other words, that R/K is J-adically complete. Note that this implies that T is a subring of S, so that if S is a domain, the same is true for T.

Definition 2.1.3. The closed immersion $\mathrm{Spf}\,T \hookrightarrow \mathrm{Spf}\,R$ is called the *scheme-theoretic image* of $f : \mathrm{Spf}\,S \to \mathrm{Spf}\,R$.

We present now some technical lemmas on formal schemes over κ that we will use later on. We would like to thank Brian Conrad and Sean Cotner for suggesting the proofs we propose here (although any errors are due to the authors).

Lemma 2.1.4. *Suppose that* $\mathrm{Spf}\,R$ *and* $\mathrm{Spf}\,S$ *are endowed with an action of an affine group scheme* G/κ *and let* $f : \mathrm{Spf}\,S \to \mathrm{Spf}\,R$ *be a* G*-equivariant morphism of formal schemes over* κ*. There is a unique action of* G *on the scheme-theoretic image* $\mathrm{Spf}\,T$ *such that the induced maps* $\mathrm{Spf}\,T \to \mathrm{Spf}\,R$ *and* $\mathrm{Spf}\,S \to \mathrm{Spf}\,T$ *are* G*-equivariant.*

Proof. Let $C = \Gamma(S)$ be the coordinate ring of G, then there is a commutative diagram (all the tensor products are over κ)

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R/K \arrow[r] \arrow[d, dotted] & S \arrow[d] \\ R\widehat{\otimes}C \arrow[r] & R/K\widehat{\otimes}C \arrow[r] & S\widehat{\otimes}C, \end{tikzcd}$$

where K is the kernel of $f : R \to S$. We are trying to show that there is a unique way to fill in the dotted arrow, which will follow from the following claim.

Claim 2.1.5. *The map* $R/K\widehat{\otimes}C \to S\widehat{\otimes}C$ *is injective.*

Showing that the induced map $\operatorname{Spf} R/K = T \to G \times T$ satisfies the axioms of a group action is straightforward. $\square$

Proof of Claim 2.1.5. The completed tensor product of R/K with C is the I-adic completion of the usual tensor product $R/K \otimes C$, where I is an ideal of definition of R (and thus also an ideal of definition of R/K). As a module over κ we know that $C = k^{\oplus \mathscr{A}}$ for some index set $\mathscr{A}$ and since tensor products commute with direct sums we see that $R/K \otimes C = (R/K)^{\oplus \mathscr{A}}$. The $\mathfrak{m}_R$-adic completion of this tensor product is the subset of the product of R/K over $\mathscr{A}$, consisting of those sequences where for all $m \geq 0$ almost all elements of the sequence lie in I^m.

Let J be an ideal of definition of S containing $f(I)$, then the completion $S \widehat{\otimes} C$ is the J-adic completion of $S \otimes C$. This can be identified with the subset of the product of S over $\mathscr{A}$, consisting of those sequences where for all $m \geq 0$ almost all elements of the sequence lie in J^m. Since $R/K \to S$ is injective we see that the map

$$\prod_{\mathscr{A}} R/K \to \prod_{\mathscr{A}} S$$

is injective and hence the map $R/K \widehat{\otimes} C \to S \widehat{\otimes} C$ is injective. $\square$

Lemma 2.1.6. *Let $(A, \mathfrak{m}), (B, \mathfrak{n})$ be Noetherian complete local rings over κ, let C, D over κ be adic rings and let $A \to C$ and $B \to D$ be adic and injective maps of κ-algebras. Then the induced map (where the tensor products are over κ)*

$$A \widehat{\otimes} B \to C \widehat{\otimes} D$$

is injective.

Proof. The map of A-modules

$$A \otimes B \to C \otimes D$$

is injective because we are working over a field. For $j \geq 0$ let $I_j = \mathfrak{m}_A^j C$, then the topology on C is defined by the chain of ideals I_j and since the topology on C is Hausdorff by assumption we find that $\bigcap_j I_j = (0)$. This means that we similarly have $\bigcap_j (I_j \cap A) = 0$ and because A is Noetherian, Chevalley's theorem (see exercise 8.7 of [58] and its solution on page 290) says that for each j there is an $n(j)$ such that $(I_{n(j)} \cap A) \subseteq \mathfrak{m}_A^j$. By construction $\mathfrak{m}_A^j \subseteq (I_j \cap A)$ and so we have shown that the $\mathfrak{m}_A$-adic topology on A is equal to the linear topology induced by the chain of ideals $(I_1 \cap A) \supseteq (I_2 \cap A) \supseteq \cdots$. In other words,

$$A = \varprojlim_j A/(I_j \cap A),$$

as a topological κ-vector space, and similarly

$$B = \varprojlim_j B/(J_j \cap B)$$

where $J_j = \mathfrak{m}_B^k D$. By construction the maps $A/(I_j \cap A) \to C/I_j$ and $B/(J_j \cap B) \to D/J_j$ are injective. It follows that for each (j, j') the map

$$A/(I_j \cap A) \otimes B/(J_{j'} \cap B) \to C/I_j \otimes D/J_{j'}$$

is injective because it factors as a composition of the maps

$$A/(I_j \cap A) \otimes B/(J_{j'} \cap B) \to A/I_j \otimes D/J_{j'} \to C/I_j \otimes D/J_{j'}$$

which are injective because we are over a field. By the definition of the completed tensor product the map that we are trying to show is injective is

$$\varprojlim_{j,j'} \left(A/(I_j \cap A) \otimes B/(J_{j'} \cap B) \right) \to \varprojlim_{j,j'} \left(C/I_j \otimes D/J_{j'} \right).$$

This is an inverse limit of injective maps and thus injective. $\square$

Lemma 2.1.7. *Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be complete (Noetherian) local rings over κ with residue fields isomorphic to κ. Let $f : \operatorname{Spf} B \to \operatorname{Spf} A$ be a monomorphism of formal schemes over κ, then it is a closed immersion.*

Proof. We have a local homomorphism of κ-algebras $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ that induces an isomorphism on residue fields and such that for every Artinian local ring R the induced map $\operatorname{Hom}(B, R) \to \operatorname{Hom}(A, R)$ is injective.

Let $R = B/(\mathfrak{m}B + \mathfrak{n}^j)$ and assume for the sake of contradiction that $\mathfrak{m}B + \mathfrak{n}^j$ is strictly contained in $\mathfrak{n}$. Then the natural map $B \to R$ does not factor through $B/\mathfrak{n}$. Note that there is another natural map $B \to R$ given by

$$B \to B/\mathfrak{n} = A/\mathfrak{m} \to B/\mathfrak{m}B \to R,$$

and these two natural maps are different by our assumption. But they agree after precomposition with A, so this contradicts the injectivity of $\operatorname{Hom}(B, R) \to \operatorname{Hom}(A, R)$, and thus $\mathfrak{m}_B + \mathfrak{n}^j = \mathfrak{n}$.

Lemma 0AMS of [73] then tells us that the closure of the ideal $\mathfrak{m}B$ is equal to $\mathfrak{n}$ and it follows that the closure of the ideal $\mathfrak{m}^j B$ is equal to $\mathfrak{n}^j$. Therefore the map f is taut and thus by Lemma 0APU of [73] the topology on B is given by the $\mathfrak{m}$-adic topology. Thus $B/\mathfrak{m}_B^j$ has the discrete topology and it follows that the following diagram is Cartesian

$$\begin{tikzcd} \operatorname{Spec} B/\mathfrak{m}_B^j \arrow[r] \arrow[d] & \operatorname{Spf} B \arrow[d] \\ \operatorname{Spec} A/\mathfrak{m}^j \arrow[r] & \operatorname{Spf} A. \end{tikzcd}$$

This tells us that $f : \operatorname{Spf} B \to \operatorname{Spf} A$ is representable, and Lemma 0GHZ of [73] allows us to conclude that f is a closed immersion. $\square$

2.2. **p-divisible groups and Dieudonné-theory.** Let R be a semiperfect $\mathbb{F}_p$-algebra and let $A_{\mathrm{cris}}(R)$ be Fontaine's ring of crystalline periods (see [68, Prop. 4.1.3]) with $\varphi : A_{\mathrm{cris}}(R) \to A_{\mathrm{cris}}(R)$ induced by the absolute Frobenius on R.

Definition 2.2.1. A *Dieudonné module* over R is a pair (M^+, φ_{M^+}), where M^+ is a finite locally free $A_{\mathrm{cris}}(R)$-module and where

$$\varphi_{M^+} : \varphi^* M^+[\tfrac{1}{p}] \to M^+[\tfrac{1}{p}]$$

is an isomorphism such that

$$M^+ \subseteq \varphi_{M^+}(M^+) \subseteq \tfrac{1}{p} M^+.$$

Remark 2.2.2. Usually one instead asks that

$$pM^+ \subseteq \varphi_{M^+}(M^+) \subseteq M^+;$$

our conventions agree, for example, with the ones in [9]. A p-divisible group X over R has a covariant Dieudonné-module $(\mathbb{D}(X), \varphi_X)$, normalised as in [9]. In particular this means that the Dieudonné-module of $\mathbb{Q}_p/\mathbb{Z}_p$ over R is $A_{\mathrm{cris}}(R)$ equipped with the trivial Frobenius, and the Dieudonné-module of μ_{p^∞} is $A_{\mathrm{cris}}(R)$ equipped with Frobenius given by $1/p$. This also means that the contravariant Dieudonné module is isomorphic to the dual of the covariant Dieudonné module, see [9, footnote on page 692].

Dieudonné-modules are particularly well behaved when R is semiperfect and *quasisyntomic.*

Definition 2.2.3. We say that X is *quasisyntomic* if the relative cotangent complex $L\Omega^1_{X/\mathbb{F}_p}$ has Tor-amplitude in $[-1,0]$.

Definition 2.2.4. We say that $X \to Y$ is a *quasisyntomic cover* if the relative cotangent complex $L\Omega^1_{X/Y}$ has Tor-amplitude in $[-1,0]$ and $X \to Y$ is an fpqc cover.

If X and X' are p-divisible groups over R then there is a natural map

$$\mathrm{Hom}_R(X, X') \to \mathrm{Hom}_\varphi(\mathbb{D}(X), \mathbb{D}(X')),$$

where the right hand side denotes homomorphisms of $A_{\mathrm{cris}}(R)$-modules that intertwine φ_X and $\varphi_{X'}$. Theorem 4.8.5 of [1] tells us that this natural map is an isomorphism if R is quasisyntomic and semiperfect. Note that the theorem is stated for contravariant Dieudonné-theory with the usual normalisation. The result that we mention follows by taking the dual.

Definition 2.2.5 (cf. Definition 1.1.4 of [1])**.** We call an $\mathbb{F}_p$-scheme *qrsp* (quasi-regular semiperfect) if it is quasisyntomic and semiperfect.

The main example is

$$\operatorname{Spec} \kappa[X_1^{1/p^\infty}, \cdots, X_m^{1/p^\infty}]/(X_1, \cdots, X_m)$$

where κ is a perfect field.

2.2.6. We recall that a connected p-divisible group Y over $\overline{\mathbb{F}}_p$, considered as a functor on $\overline{\mathbb{F}}_p$-algebras, is representable by the formal spectrum of a power series ring over $\overline{\mathbb{F}}_p$ (see e.g. Lemma 3.1.1 of [68] for a more general result). The scheme-theoretic p-adic Tate-module

$$T_p Y := \varprojlim_n Y[p^n]$$

where the transition maps are given by multiplication by p, is representable by an affine scheme. Proposition 4.12.(4) of [9] tells us that this affine scheme is isomorphic to the spectrum of

$$\overline{\mathbb{F}}_p[X_1^{1/p^\infty}, \cdots, X_m^{1/p^\infty}]/(X_1, \cdots, X_m)$$

for some m; note that this ring is qrsp. If Y is connected then the universal cover

$$\tilde{Y} := \varprojlim Y,$$

where the transition maps are given by multiplication by p, is representable by the formal spectrum of the completed perfection of a power series ring by Proposition 3.1.3.(iii) of [68]. Therefore $\tilde{Y}$ is a filtered colimit of spectra of semiperfect rings and so it is determined by its values on semiperfect rings.

2.3. **Frobenius-smooth schemes and the Tannakian category of isocrystals.** We start with a definition.

Definition 2.3.1. We say that a scheme X over $\mathbb{F}_p$ is *Frobenius-smooth* if $F : X \to X$ is syntomic.

By Lemma 2.1.1.(2) of [26] this is equivalent to X being Zariski locally of the form $\operatorname{Spec} B$ where B has a finite (absolute) p-basis. In other words there exist elements $x_1, \cdots, x_n$ such that every element $b \in B$ can be uniquely written as

$$b = \sum b_\alpha^p x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where $0 \leq \alpha_i \leq p-1$ and where $b_\alpha \in B$. The main examples of Frobenius-smooth schemes that we will encounter are smooth schemes over perfect fields and power series rings over perfect fields. If X is a Noetherian Frobenius-smooth scheme, then it is regular by a result of Kunz (see Lemma 0EC0 of [73]), thus in particular quasisyntomic.

Definition 2.3.2. If X is a scheme over $\mathbb{F}_p$ we denote by $\operatorname{Isoc}(X)$ the category of crystals of finitely generated quasi-coherent $\mathcal{O}$-modules on the absolute crystalline site of X with Hom-sets tensored by $\mathbb{Q}$.

Following [26], we define

$$\kappa := \bigcap_{i=1}^{\infty} \Gamma(X, \mathcal{O}_X)^{p^n}.$$

to be the *ring of constants* of X and we write K for $W(\kappa)[\frac{1}{p}]$. Note that if X is a geometrically connected scheme of finite type over a perfect field then κ coincides with the base field.

Proposition 2.3.3 ([26, Corollary 3.3.3])**.** *If X is an irreducible Noetherian Frobenius-smooth scheme, then* $\operatorname{Isoc}(X)$ *is a K-linear Tannakian category.*

This allows us to define the monodromy groups of isocrystals in this situation.

Definition 2.3.4. Let X be a scheme satisfying the assumptions in Proposition 2.3.3 and let $\mathcal{M}$ be an isocrystal over X. We define $\langle \mathcal{M} \rangle$ to be the Tannakian subcategory of $\operatorname{Isoc}(X)$ generated by $\mathcal{M}$. If η is an Ω-point for some perfect field Ω, we define $G(\mathcal{M}, \eta)$ to be the Tannaka group of $\langle \mathcal{M} \rangle$ with respect to the fibre functor induced by η. We call it the *monodromy group of* $\mathcal{M}$ with respect to η.

Notation 2.3.5. In Theorem II, Section 3.4, and Section 6 we also need to consider the monodromy group of the restriction of an isocrystal $\mathcal{M}$, defined on a smooth variety X over a perfect field of characteristic p, to $X^{/x} = \operatorname{Spf} \widehat{\mathcal{O}}_{X,x}$ where $x \in X$ is a closed point. For simplicity in this situation we will rather consider the restriction of $\mathcal{M}$ via the morphism $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x} \to X$, denoted by $\mathcal{M}^{/x}$. Therefore in these sections we will treat $X^{/x}$ as a scheme rather than a formal scheme. If η is a perfect point of $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ we define then $G(\mathcal{M}^{/x}, \eta)$ as the monodromy group of $\mathcal{M}^{/x}$ over $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ (note that $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ is Frobenius-smooth). Arguing as in [24, Proposition 2.4.8] one can show that this monodromy group is the same as the one constructed considering the restriction to the formal scheme $\operatorname{Spf} \widehat{\mathcal{O}}_{X,x}$.

2.3.6. Both $\tilde{Y}$ and Y are determined by their restriction to the category of semiperfect $\overline{\mathbb{F}}_p$-algebras, and we can describe them explicitly on the category of qrsp $\overline{\mathbb{F}}_p$-algebras as follows.

Lemma 2.3.7. *There is a commutative diagram of natural transformation of functors on the category of qrsp $\overline{\mathbb{F}}_p$-algebras, which evaluated at an object R gives*

$$\begin{tikzcd} T_pY(R) \arrow[r, "\simeq"] \arrow[d, hook] & \left(A_{\mathrm{cris}}(R)\otimes_{\breve{\mathbb{Z}}_p}\mathbb{D}(Y)\right)^{\varphi=1} \arrow[d] \\ \tilde{Y}(R) \arrow[r, "\simeq"] & \left(B^+_{\mathrm{cris}}(R)\otimes_{\breve{\mathbb{Q}}_p}\mathbb{D}(Y)[\frac{1}{p}]\right)^{\varphi=1} \end{tikzcd},$$

where φ is given by the diagonal Frobenius and where $B^+_{\mathrm{cris}}(R) = A_{\mathrm{cris}}(R)[1/p]$.

Proof. Let R be qrsp, then Theorem 4.8.5 of [1] tells us that Dieudonné-module functor gives us a natural isomorphism

$$\begin{aligned} T_pY(R) &= \mathrm{Hom}_R((\mathbb{Q}_p/\mathbb{Z}_p)_R, Y_R) \\ &\to \mathrm{Hom}_{A_{\mathrm{cris}},\varphi}(A_{\mathrm{cris}}(R), A_{\mathrm{cris}}(R)\otimes_{\breve{\mathbb{Z}}_p}\mathbb{D}(Y)) \\ &\simeq \left(A_{\mathrm{cris}}(R)\otimes_{\breve{\mathbb{Z}}_p}\mathbb{D}(Y)\right)^{\varphi=1} \end{aligned}$$

where the latter bijection is induced by evaluation at 1. Similarly after inverting p we get a natural isomorphism

$$\begin{aligned} \tilde{Y}(R) &= \mathrm{Hom}_R((\mathbb{Q}_p/\mathbb{Z}_p)_R, Y_R)[\tfrac{1}{p}] \\ &\to \left(B^+_{\mathrm{cris}}(R)\otimes_{\breve{\mathbb{Q}}_p}\mathbb{D}(Y)[\tfrac{1}{p}]\right)^{\varphi=1} \end{aligned}$$

and the diagram commutes by construction. □

3. Local monodromy of F-isocrystals

The main goal of this section is to prove Theorem II. As it often happens, to show the relation between the two Tannakian groups in the statement we first find an equivalent categorical condition. We do this in Section 3.1, where we prove a quite general Tannakian criterion to check that a unipotent subgroup of an algebraic group is the entire unipotent radical.

After that, in Section 3.2, we prove that the global monodromy group of an F-isocrystal with constant Newton polygon is the same as its "generic" monodromy group (Theorem 3.2.5). This will be essential to reduce the entire problem to a problem of F-isocrystals defined over (imperfect) fields with finite p-basis. In Section 3.3 we prove then some descent results, notably the descent of splittings of the slope filtration for separable field extensions with finite p-basis (Proposition 3.3.5), and in Section 3.4 we put all the ingredients together and we prove Theorem II.

3.1. **A Tannakian criterion.** Let K be a field and let V be a finite-dimensional K-vector space.

Lemma 3.1.1. *If U, U' are unipotent subgroups of $\mathrm{GL}(V)$, then the following two properties are equivalent.*

(i) *$U' \subseteq U$.*

(ii) *$W^U \subseteq W^{U'}$ for every algebraic representation W of $\mathrm{GL}(V)$.*

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. To prove (ii) $\Rightarrow$ (i) we just note that by Chevalley's theorem there is a representation W of $\mathrm{GL}(V)$ such that U is the stabiliser of a line $L \subseteq W$. Since U does not admit non-trivial characters, we deduce that $L \subseteq W^U$. By (ii) this implies that U' fixes L and this yields the desired result. $\square$

Thanks to this lemma, we get a Tannakian criterion to prove that a unipotent subgroup of an algebraic group coincides with the unipotent radical. This will be the criterion that we will use to prove Theorem II.

Proposition 3.1.2. *Let $U \subseteq G \subseteq \mathrm{GL}(V)$ be a chain of subgroups of $\mathrm{GL}(V)$ and suppose that there exists a cocharacter $\nu : \mathbb{G}_m \to \mathrm{GL}(V)$ such that both U and $R_u(G)$ are in U_ν. The following two properties are equivalent.*

(i) *$U = R_u(G)$.*

(ii) *For every representation W of $\mathrm{GL}(V)$, the group G stabilises W^U and the induced representation factors through $G/R_u(G)$.*

Proof. (1) $\Rightarrow$ (2) follows from the observation that $R_u(G)$ is normal in G. For (2) $\Rightarrow$ (1) first note that by the assumptions $U \subseteq R_u(G)$ since $R_u(G) = G \cap U_\nu$. For the other inclusion, thanks to (2) we deduce that for every representation W we have that $W^U \subseteq W^{R_u(G)}$. Thus by Lemma 3.1.1, we conclude that $R_u(G) \subseteq U$. $\square$

3.2. **Passing to the generic point.** Let A_0 be an $\mathbb{F}_p$-algebra and let $A_\infty = \varinjlim_{i\in I} A_i$ be a filtered colimit of A_0-algebras, where A_0 is the initial object of the system. We write I' for $I \cup \{\infty\}$ and for a crystal in quasi-coherent modules $\mathcal{M}_0$ over $\operatorname{Spec} A_0/\mathbb{Z}_p$ we consider for every $i \in I'$ the base changes of $\mathcal{M}_0$ to $\mathrm{Cris}(\operatorname{Spec} A_i/\mathbb{Z}_p)$, denoted by $\mathcal{M}_i$.

Lemma 3.2.1. *For every $j \geq 0$, we have*

$$H^j(\mathrm{Cris}(\operatorname{Spec} A_\infty/\mathbb{Z}_p), \mathcal{M}_\infty) = \left(\varinjlim_{i\in I} H^j(\mathrm{Cris}(\operatorname{Spec} A_i/\mathbb{Z}_p), \mathcal{M}_i)\right)^\wedge.$$

Proof. For $i \in I'$, let P_i be the free commutative $\mathbb{Z}_p$-algebra associated to the set underlying A_i, so that $P_\infty = \varinjlim_{i\in I} P_i$. We write $P_i(n)$ (resp. $A_i(n)$) for the tensor product of $n+1$ copies of P_i (resp. A_i) over $\mathbb{Z}_p$ (resp. $\mathbb{F}_p$), and $J_i(n)$ for the kernel of $P_i(n) \to A_i(n)$. We also write $D_i(n)$ for the p-adic completion of the divided power envelope of $P_i(n)$ with respect to the ideal $J_i(n)$ and for every $i \in I'$, $n \geq 0$, and $e > 0$ we write $M_{i,e}(n)$ for

$$\mathcal{M}_i(A_i(n), D_i(n)/p^e, \gamma_{i,e}(n)),$$

where $\gamma_{i,e}(n)$ is the natural divided power structure on $J_i(n)/p^e$. The cohomology of $\mathcal{M}_i$ is computed as the cohomology of the complex $\varprojlim_e M_{i,e}(\bullet)$. Since $D_\infty(n)/p^e = \varinjlim_{i\in I} D_i(n)/p^e$, we deduce that for each $e > 0$ we have that $M_{\infty,e}(\bullet) = \varinjlim M_{i,e}(\bullet)$. This yields the desired result. $\square$

Proposition 3.2.2. *Let X be an irreducible Noetherian Frobenius-smooth scheme over $\mathbb{F}_p$ with generic point η. For every isocrystal $\mathcal{M}$ over X, we have*

$$H^0(\eta, \mathcal{M}_\eta) = \varinjlim_{U\subseteq X} H^0(U, \mathcal{M}_U),$$

where the colimit is over the dense open subschemes of X.

Proof. By Lemma 3.2.1, we have that $H^0(\eta, \mathcal{M}_\eta) = \left(\varinjlim_{U\subseteq X} H^0(U, \mathcal{M}_U)\right)^\wedge$. Let κ be the field of constants of X. We note that κ is the field of constants of every $U \subseteq X$ and η. Therefore, if K is the fraction field of the ring of Witt vectors of κ, each group $H^0(U, \mathcal{M}_U)$ is a K-linear subspace of $H^0(\eta, \mathcal{M}_\eta)$. Since η is Frobenius-smooth, this implies that $\varinjlim_{U\subseteq X} H^0(U, \mathcal{M}_U)$ is a finite dimensional K-vector space as well. In particular, it is already p-adically complete. This ends the proof. □

Suppose now that X is a smooth irreducible variety over a perfect field with generic point η.

Proposition 3.2.3. *If $\mathcal{N}$ is a subquotient of an isocrystal $\mathcal{M}$ over X such that $F^*\mathcal{M} \simeq \mathcal{M}$, then $H^0(\eta, \mathcal{N}_\eta) = H^0(X, \mathcal{N})$.*

Proof. By Theorem 5.10 of [28], we know that $\mathcal{N}$ is a subobject of some isocrystal $\mathcal{M}'$ such that $F^*\mathcal{M}' \simeq \mathcal{M}'$. Thanks to [*ibid.*, Lemma 5.6] it is enough to prove the result for $\mathcal{M}'$. By [27, Theorem 2.2.3], for every dense open $U \subseteq X$, we have that $H^0(X, \mathcal{M}) = H^0(U, \mathcal{M}_U)$. Therefore, thanks to Proposition 3.2.2 we deduce that $H^0(\eta, \mathcal{M}'_\eta) = H^0(X, \mathcal{M}')$. □

Proposition 3.2.4. *If $(\mathcal{N}_\eta, \Phi_{\mathcal{N}_\eta})$ is an isoclinic subobject of $(\mathcal{M}_\eta, \Phi_{\mathcal{M}_\eta})$, then $\mathcal{N}_\eta \subseteq \mathcal{M}_\eta$ extends to a subobject of $\mathcal{M}$.*

Proof. It is enough to prove the result after replacing $\Phi_{\mathcal{M}}$ with some power and after twisting $(\mathcal{M}, \Phi_{\mathcal{M}})$ by a rank 1 F-isocrystal. Therefore we may assume that $(\mathcal{N}_\eta, \Phi_{\mathcal{N}_\eta})$ is a unit-root F^n-isocrystal. We have that $(\mathcal{N}_\eta, \Phi_{\mathcal{N}_\eta})$ is a subobject of the unit-root part of the graded object associated to the slope filtration of $(\mathcal{M}_\eta, \Phi_{\mathcal{M}_\eta})$, denoted by $(\mathcal{N}'_\eta, \Phi_{\mathcal{N}'_\eta})$. By [2, Theorem 6], the F^n-isocrystals $(\mathcal{N}_\eta, \Phi_{\mathcal{N}_\eta})$ and $(\mathcal{N}'_\eta, \Phi_{\mathcal{N}'_\eta})$ correspond to lisse $\mathbb{Q}_{p^n}$-sheaves V and V' over η. Since $(\mathcal{M}, \Phi_{\mathcal{M}})$ admits the slope filtration, V' extends to a lisse sheaf over X. The lisse sheaf V, being a subobject of V', extends as well over X. This implies that $(\mathcal{N}_\eta, \Phi_{\mathcal{N}_\eta})$ extends to an F^n-isocrystal $(\mathcal{N}, \Phi_{\mathcal{N}})$ over X. Write $\mathcal{H}$ for the isocrystal $\underline{\mathrm{Hom}}(\mathcal{N}, \mathcal{M})$. □

Theorem 3.2.5. *For every F-isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ over X with constant Newton polygon we have that $\langle \mathcal{M} \rangle \to \langle \mathcal{M}_\eta \rangle$ is an equivalence of Tannakian categories. In other words, $G(\mathcal{M}, \eta^{\mathrm{perf}}) = G(\mathcal{M}_\eta, \eta^{\mathrm{perf}})$.*

Proof. The natural functor $\langle \mathcal{M} \rangle \to \langle \mathcal{M}_\eta \rangle$ is a functor between Tannakian categories. Thanks to Proposition A.3 and A.4.1 of [28], to show that it is an equivalence we have to prove that it is fully faithful and that every rank 1 object $\mathcal{L}_\eta \in \langle \mathcal{M}_\eta \rangle$ is a direct summand of a semi-simple isocrystal coming from $\langle \mathcal{M} \rangle$. For the first part it is enough to prove that the functor induces isomorphisms at the level of global sections, which follows from Proposition 3.2.3. To prove the second part we note that a rank 1 isocrystal $\mathcal{L}_\eta \in \langle \mathcal{M}_\eta \rangle$ is the subquotient of some isocrystal $\mathcal{M}'_\eta \in \langle \mathcal{M}_\eta \rangle$ coming from $\langle \mathcal{M} \rangle$ which can be endowed with a Frobenius structure with constant Newton polygon. Also, taking the subquotient of $\mathcal{M}'_\eta$ associated to some slope, we may assume that $\mathcal{L}_\eta$ is actually a subobject of $\mathcal{M}'_\eta$ and that the latter isocrystal can be endowed with an isoclinic Frobenius structure. Write $\mathcal{N}_\eta \subseteq \mathcal{M}'_\eta$ for the sum of rank 1 subobjects of $\mathcal{M}'_\eta$. By construction, $\mathcal{N}_\eta \subseteq \mathcal{M}'_\eta$ is kept stable by the Frobenius structure of $\mathcal{M}'_\eta$. By virtue of Proposition 3.2.4, this implies that isocrystal $\mathcal{N}_\eta$ comes from an isocrystal $\mathcal{N} \in \langle \mathcal{M} \rangle$. This ends the proof. □

3.3. **Descent for isocrystals.** We prove now various descent results that we will need in the next section. Let $f : Y \to X$ be a pro-étale Π-cover of Noetherian Frobenius-smooth schemes over $\mathbb{F}_p$, where Π is a profinite group and let $y \in Y(\Omega)$ be an Ω-point of Y with Ω a perfect field. Write K for the fraction field of the ring of Witt vectors of Ω.

Lemma 3.3.1. *For every isocrystal $\mathcal{M}$ over X, the maximal trivial subobject of $f^*\mathcal{M}$ descends to a subobject $\mathcal{N} \subseteq \mathcal{M}$. Moreover, if $\mathcal{M}$ is endowed with a Frobenius structure $\Phi_{\mathcal{M}}$, the inclusion $\mathcal{N} \subseteq \mathcal{M}$ upgrades to an inclusion $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (\mathcal{M}, \Phi_{\mathcal{M}})$ of F-isocrystals and $(\mathcal{N}, \Phi_{\mathcal{N}})$ is a direct sum of isoclinic F-isocrystals.*

Proof. Since the cover $Y \to X$ is a quasisyntomic cover, it satisfies descent for isocrystals thanks to Proposition 3.5.4 of [26] (see also [57] or Section 2 of [5]). By the assumption,

$$Y \times_X Y \simeq \varprojlim_{U \subseteq \Pi} (Y \times_X Y)^U$$

where the limit runs over all the open normal subgroups of Π and $(Y \times_X Y)^U := \coprod_{[\gamma] \in \Pi/U} Y_{[\gamma]}$ is a disjoint union of copies of Y. The group Π acts on $Y \times_X Y$ in the obvious way. Since $f^*\mathcal{M}$ comes from X, it is endowed with a descent datum with respect to the cover $Y \to X$. This datum consists of isomorphisms $\gamma^*\mathcal{M}_{(Y \times_X Y)^U} \simeq \mathcal{M}_{(Y \times_X Y)^U}$ for each $U \subseteq \Pi$ and $\gamma \in \Pi$. The functor γ^* sends trivial objects to trivial objects, which implies that the descent datum restricts to a descent datum of $\mathcal{T}$, the maximal trivial subobject of $f^*\mathcal{M}$. Therefore, $\mathcal{T}$ descends to a subobject $\mathcal{N} \subseteq \mathcal{M}$, as we wanted. If $\mathcal{M}$ is endowed with a Frobenius structure, then it induces a Frobenius structure on each isocrystal $\mathcal{M}_{(Y \times_X Y)^U}$ and this structure preserves each maximal trivial subobject of given slope. This implies that even the descended object $\mathcal{N} \subseteq \mathcal{M}$ is stabilised by the Frobenius and the induced Frobenius structure satisfies the desired property. □

Proposition 3.3.2. *Let $(\mathcal{M}, \Phi_{\mathcal{M}})$ be an F-isocrystal which admits the slope filtration and write ν for the associated Newton cocharacter. If $R_u(G(\mathcal{M}, f(y))) \subseteq U_\nu$ and $\mathrm{Gr}_{S_\bullet}(f^*\mathcal{M})$ is trivial, then $G(f^*\mathcal{M}, y) = R_u(G(\mathcal{M}, f(y)))$.*

Proof. Since $\mathrm{Gr}_{S_\bullet}(f^*\mathcal{M})$ is trivial, the group $G(f^*\mathcal{M}, y)$ is a unipotent subgroup of $G(\mathcal{M}, f(y)) \otimes_{K'} K$ sitting inside U_ν. Therefore, we are in the situation of Proposition 3.1.2 and we have to prove that (ii) is satisfied. This amounts to show that for every $m, n \geq 0$, the maximal trivial subobject $\mathcal{T} \subseteq f^*(\mathcal{M}^{\otimes m} \otimes (\mathcal{M}^\vee)^{\otimes n})$ descends to a semi-simple isocrystal over X. By Lemma 3.3.1, we know that $\mathcal{T}$ descends to an isocrystal $\mathcal{N}$ which is the direct sum of isocrystals which can be endowed with an isoclinic Frobenius structure. Since $R_u(G(\mathcal{M}, f(y)))$ is contained in U_ν, we deduce that $\mathcal{N}$ is semi-simple, as we wanted. □

Lemma 3.3.3. *If k'/k is a separable field extension and k' admits a finite p-basis, then $k' \otimes_k k'$ admits a finite p-basis as well.*

Proof. Thanks to [58, Theorem 26.6], the field k admits a finite p-basis $t_1, \cdots, t_d$ which extends to a finite p-basis $t_1, \cdots, t_d, u_1, \cdots u_e$ of k'. We claim that $\Gamma := \{t_i \otimes 1\}_i \cup \{u_i \otimes 1\}_i \cup \{1 \otimes u_i\}_i$ is a finite p-basis of $k' \otimes_k k'$. It is clear from the construction that the elements of Γ generate $k' \otimes_k k'$ over $(k' \otimes_k k')^p$. On the other hand, the exact sequence

$$0 \to \Omega^1_{k/\mathbb{F}_p} \otimes_k (k' \otimes_k k') \to \Omega^1_{k' \otimes_k k'/\mathbb{F}_p} \to (\Omega^1_{k'/k} \otimes_k k') \oplus (k' \otimes_k \Omega^1_{k'/k}) \to 0$$

shows that the elements $d\gamma$ with $\gamma \in \Gamma$ form a basis of the free module $\Omega^1_{k' \otimes_k k'/\mathbb{F}_p}$. We deduce the p-independence of the elements of Γ by arguing as in Lemma 07P2 of [73]. □

Lemma 3.3.4. *If X is a Frobenius-smooth scheme over $\mathbb{F}_p$, every locally free F-isocrystal with constant Newton polygon which does not admit slope 0 has no global sections fixed by the Frobenius structure.*

Proof. Since X is Frobenius-smooth, the global sections of any isocrystal over X embed into the global sections of the base change to X^{perf}. Over X^{perf} one can argue as in the proof of [46, Theorem 2.4.2]. We reduce to the case when $X^{\mathrm{perf}} = \operatorname{Spec} A$ with A perfect. Then we can embed A into a product of perfect fields. This reduces the problem to the case of perfect fields where the result is well-known. □

Proposition 3.3.5. *Let $k \subseteq k'$ a separable extension of characteristic p fields with finite p-basis and let $(\mathcal{M}, \Phi_{\mathcal{M}})$ be a free F-isocrystal over k with slope filtration $S_\bullet$ of length n. If $\mathcal{M}_{k'}$ admits a Frobenius-stable splitting of the form $\mathcal{N}_{k'} \oplus S_{n-1}(\mathcal{M}_{k'})$ with $\mathcal{N}_{k'}$ some subobject of $\mathcal{M}_{k'}$, the same is true for $\mathcal{M}$.*

Proof. Since $\operatorname{Spec} k' \to \operatorname{Spec} k$ is a quasisyntomic cover, it satisfies descent for isocrystals thanks to the descent results of Drinfeld and Mathew in [26, 57] (see Theorem 2.2 of [5]). Therefore, in order to descend $\mathcal{N}_{k'}$ to k it is enough to show that the splitting $\mathcal{N}_{k'\otimes_k k'} \oplus S_{n-1}(\mathcal{M}_{k'\otimes_k k'})$ is unique. Suppose that $\mathcal{N}'_{k'\otimes_k k'} \oplus S_{n-1}(\mathcal{M}_{k'\otimes_k k'})$ was a different splitting. Then there would exist a non-trivial Frobenius-equivariant morphism $\mathcal{N}'_{k'\otimes_k k'} \to S_{n-1}(\mathcal{M}_{k'\otimes_k k'})$. In other words, the F-isocrystal $\underline{\mathrm{Hom}}(\mathcal{N}'_{k'\otimes_k k'}, S_{n-1}(\mathcal{M}_{k'\otimes_k k'}))$ would have a non-trivial Frobenius-invariant global section. Since the slopes of $\underline{\mathrm{Hom}}(\mathcal{N}'_{k'\otimes_k k'}, S_{n-1}(\mathcal{M}_{k'\otimes_k k'}))$ are all negative by definition and $k' \otimes_k k'$ admits a finite p-basis by Lemma 3.3.3, this would contradict Lemma 3.3.4. □

3.4. **The local monodromy theorem.** We are ready to put all the previous results together and prove Theorem II. Let X be a smooth irreducible variety over a perfect field and let x be a closed point of X. In this section we view $X^{/x}$ as the scheme $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ (conventions of Notation 2.3.5 are in force). We denote by k the function field of X and by k_x the function field of $X^{/x}$. We also write η^{sep} (resp. $\bar{\eta}$) for the points over the generic point of X associated to a separable (resp. algebraic) closure of k.

Lemma 3.4.1. *The fields k and k_x have a common finite p-basis. In particular, $k \subseteq k_x$ is a separable field extension.*

Proof. By [58, Theorem 26.7], it is enough to show that $\Omega^1_{k/\mathbb{F}_p} \otimes_k k_x = \Omega^1_{k_x/\mathbb{F}_p}$. Write A for the local ring of X at x and $A_x^\wedge$ for the completion with respect to the maximal ideal $\mathfrak{m}_x$. Since A is regular, thanks to Theorem 30.5 and Theorem 30.9 of [58], we deduce that $\Omega^1_{A/\mathbb{F}_p} \otimes_A A_x^\wedge = \Omega^1_{A_x^\wedge/\mathbb{F}_p}$. We get the desired result after inverting $\mathfrak{m}_x - \{0\}$. □

Proposition 3.4.2. *If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is an F-isocrystal over X such that $R_u(G(\mathcal{M}, \bar{\eta})) \subseteq U_\nu$, then $G(\mathcal{M}_{\eta^{\mathrm{sep}}}, \bar{\eta}) = R_u(G(\mathcal{M}, \bar{\eta}))$.*

Proof. By Theorem 3.2.5 we have that $G(\mathcal{M}, \bar{\eta}) = G(\mathcal{M}_\eta, \bar{\eta})$, so that we are reduced to prove the statement for $G(\mathcal{M}_\eta, \bar{\eta})$. Note that the cover $f : \eta^{\mathrm{sep}} \to \eta$ is a pro-étale $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-cover and $\mathrm{Gr}_{S_\bullet}(f^*\mathcal{M}_\eta)$ is trivial because η^{sep} is simply connected. This shows that we can apply Proposition 3.3.2 and deduce the desired result. □

Proposition 3.4.3. *If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is an F-isocrystal over X coming from an irreducible overconvergent F-isocrystal with constant Newton polygon, then $H^0(X^{/x}, (S_1(\mathcal{M}))^{/x}) = H^0(X^{/x}, \mathcal{M}^{/x})$.*

Proof. By Galois descent we may assume that the ring of constants of X is an algebraically closed field. The inclusion $H^0(X^{/x}, (S_1(\mathcal{M}))^{/x}) \subseteq H^0(X^{/x}, \mathcal{M}^{/x})$ is an inclusion of F-isocrystals over κ. We suppose by contradiction that this is not an equality. Let $s_r > s_1$ be the greatest slope appearing in $H^0(X^{/x}, \mathcal{M}^{/x})$ and let v be a non-zero vector such that $\Phi^n_{\mathcal{M}^{/x}}(v) = p^{s_r n} v$ for $n \gg 0$. Write $(\tilde{\mathcal{M}}, \Phi_{\tilde{\mathcal{M}}})$ for the base change of $(\mathcal{M}, \Phi_{\mathcal{M}})$ to η^{sep}.

By the parabolicity conjecture, [23], $R_u(G(\mathcal{M}, \bar{\eta}))$ is contained in U_ν because $(\mathcal{M}, \Phi_{\mathcal{M}})$ comes from an irreducible overconvergent F-isocrystal. Proposition 3.4.2 then implies that the monodromy group $G(\tilde{\mathcal{M}}, \bar{\eta})$ is equal to $G(\mathcal{M}, \bar{\eta}) \cap U_\nu$. Therefore, the line spanned by v determines a rank 1 subobject $\tilde{\mathcal{L}} \subseteq S_r(\tilde{\mathcal{M}})/S_{r-1}(\tilde{\mathcal{M}})$ stabilised by the Frobenius. The preimage of this isocrystal in $S_r(\tilde{\mathcal{M}})$, denoted by $\tilde{\mathcal{N}}$, is kept invariant by the Frobenius and sits in an exact sequence

$$0 \to S_{r-1}(\tilde{\mathcal{M}}) \to \tilde{\mathcal{N}} \to \tilde{\mathcal{L}} \to 0.$$

Since $(\mathcal{M}, \Phi_{\mathcal{M}})$ comes from an irreducible overconvergent F-isocrystal, the sequence does not admit a Frobenius-equivariant splitting by [23, Theorem 4.1.3]. By Proposition 3.3.5, the base change of this extension to k_x^{sep} does not split as well. This leads to a contradiction since v is a vector in $H^0(X^{/x}, \mathcal{M}^{/x})$ which produces a non-trivial global section of $\tilde{\mathcal{N}} \subseteq \tilde{\mathcal{M}}$. □

We write η_x for the generic point of $X^{/x}$ and $G(\mathcal{M}^{/x}, \eta_x^{\mathrm{perf}})$ for the monodromy group of $\mathcal{M}^{/x}$ (notation as in §2.3.5) with respect to the perfection of η_x.

Theorem 3.4.4. *If $(\mathcal{M}, \Phi_{\mathcal{M}})$ comes from a semi-simple overconvergent F-isocrystal with constant Newton polygon, then*

$$G(\mathcal{M}^{/x}, \eta_x^{\mathrm{perf}}) = R_u(G(\mathcal{M}, \eta_x^{\mathrm{perf}})).$$

Proof. Write G for the group $G(\mathcal{M}, \eta)$ and V for the induced G-representation. By [23] we have that $R_u(G)$ is contained in U_ν where ν is the Newton cocharacter. Since $X^{/x}$ is geometrically simply connected we deduce that $\mathrm{Gr}_{S_\bullet}(\mathcal{M})^{/x}$ is trivial. This implies that $G(\mathcal{M}^{/x}, \eta_x^{\mathrm{perf}}) \subseteq R_u(G) \subseteq U_\nu$. Therefore, in order to apply the criterion of Proposition 3.1.2 it is enough to show that for every $\mathcal{N} \in \langle \mathcal{M} \rangle$, the space of global sections of $\mathcal{N}^{/x}$ is the same as the fibre at x of some direct sum of isoclinic subobjects of $\mathcal{N}$. To prove this, we may assume that $\mathcal{N}$ can be endowed with a Frobenius structure $\Phi_{\mathcal{N}}$ and $(\mathcal{N}, \Phi_{\mathcal{N}})$ is irreducible. Thanks to Proposition 3.4.3, we deduce that the fibre of $S_1(\mathcal{N})$ at x is the same as $H^0(X^{/x}, \mathcal{N}^{/x})$. This yields the desired result. □

4. Automorphism groups of p-divisible groups and Dieudonné–Lie algebras

The goal of this section is to define various groups of tensor preserving automorphisms and endomorphisms of p-divisible groups with G-structure, correcting some definitions from [47]. To do this, we will introduce the notion of *Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras* and we will prove various properties using this point of view. We end by studying actions of algebraic groups on nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras and their associated unipotent groups.

4.1. **Hom groups of p-divisible groups.** We closely follow Section 3 of [15] and also Section 4 of [9] and [47]. For p-divisible groups Y and Z over $\overline{\mathbb{F}}_p$, Chai and Oort construct finite group schemes

$$\underline{\mathrm{Hom}}^{\mathrm{st}}(Y[p^n], Z[p^n]) \subseteq \underline{\mathrm{Hom}}(Y[p^n], Z[p^n]),$$

where $\underline{\mathrm{Hom}}(Y[p^n], Z[p^n])$ is the sheaf of homomorphisms from $Y[p^n]$ to $Z[p^n]$. They moreover construct maps

$$\pi_n : \underline{\mathrm{Hom}}^{\mathrm{st}}(Y[p^n], Z[p^n]) \to \underline{\mathrm{Hom}}^{\mathrm{st}}(Y[p^{n+1}], Z[p^{n+1}])$$

such that (the additive group underlying)

$$\varinjlim_n \underline{\mathrm{Hom}}^{\mathrm{st}}(Y[p^n], Z[p^n])$$

is a p-divisible group $\mathcal{H}_{Y,Z}$, called the *internal Hom p-divisible group*. Its scheme-theoretic p-adic Tate module is isomorphic to the group scheme $\underline{\mathrm{Hom}}(Y,Z)$ of homomorphisms from Y to Z by Lemma 4.1.7 of [9]. Lemma 4.1.8 of [9] tells us that there is an isomorphism

$$\mathbb{D}(\mathcal{H}_{Y,Z})[\tfrac{1}{p}] = \underline{\mathrm{Hom}}(\mathbb{D}(Y)[\tfrac{1}{p}], \mathbb{D}(Z)[\tfrac{1}{p}])^{\leq 0},$$

where $(-)^{\leq 0}$ denotes the operation of taking the subspace of slope ≤ 0 of an F-isocrystal, and where

$$\underline{\mathrm{Hom}}(\mathbb{D}(Y)[\tfrac{1}{p}], \mathbb{D}(Z)[\tfrac{1}{p}]).$$

denotes the internal hom in the category of F-isocrystals. By the proof of Lemma 4.1.8 of [9] there is an isomorphism of formal group schemes

$$\widetilde{\mathcal{H}}_{Y,Z} \simeq \underline{\mathrm{Hom}}(Y,Z)[\tfrac{1}{p}],$$

where $\widetilde{\mathcal{H}}_{Y,Z}$ is the universal cover of $\mathcal{H}_{Y,Z}$ in the sense of Scholze–Weinstein (see Section 2.2).

4.1.1. We will mostly be interested in $\mathcal{H}_Y := \mathcal{H}_{Y,Y}$ for a p-divisible group Y, in which case $T_p\mathcal{H}_Y$ and $\widetilde{\mathcal{H}}_Y$ have an algebra structure and the same for their rational variants. Up to isogeny we can write $Y = \oplus_i Y_i$ as a direct sum of isoclinic p-divisible groups with slopes in increasing order and then we can write endomorphisms of $\tilde{Y}$ in matrix form to get

$$\widetilde{\mathcal{H}}_Y = \bigoplus_{i,j} \widetilde{\mathcal{H}}_{Y_i,Y_j}.$$

Corollary 4.1.10 of [9] tells us that the p-divisible groups H_{Y_i,Y_j} are zero when $i > j$, they are étale p-divisible groups when $i = j$ and connected p-divisible groups when $i < j$. This means that we get a lower triangular matrix form (see the proof of Proposition 4.2.11 of [9]), and that the connected part

$$\widetilde{\mathcal{H}}_Y^{\circ} = \bigoplus_{i<j} \widetilde{\mathcal{H}}_{Y_i,Y_j}$$

consists of nilpotent endomorphisms. The étale part is precisely the locally profinite group scheme associated to the $\mathbb{Q}_p$-algebra given by

$$\underline{\mathrm{End}}(Y)(\overline{\mathbb{F}}_p)[\tfrac{1}{p}].$$

In order to generalise to Shimura varieties, it will be more fruitful to use the commutator bracket on $\underline{\mathrm{Hom}}(Y,Y)[\frac{1}{p}]$ to equip $\widetilde{\mathcal{H}}_Y$ with the structure of a Lie algebra. To precise it is a Lie algebra over the locally profinite ring-scheme $\underline{\mathbb{Q}_p}$ associated to the topological ring $\mathbb{Q}_p$. Here if V is a topological space we use the notation $\underline{V}$ for the functor on $\overline{\mathbb{F}}_p$-schemes sending $T \mapsto \mathrm{Hom}_{\mathrm{cont}}(|T|, V)$, where $|T|$ is the topological space underlying the scheme T. The functor $\underline{V}$ is representable by a finite scheme if V is finite and discrete, and therefore also representable if T is profinite or locally profinite.

There is a descending sequence of Lie $\underline{\mathbb{Q}_p}$-algebra ideals

$$\widetilde{\mathcal{H}}_Y^{\circ} = \mathrm{Fil}^0 \supseteq \mathrm{Fil}^1 \supseteq \mathrm{Fil}^2 \supseteq \cdots$$

where Fil^k consists of those endomorphisms that lie in

$$\bigoplus_{j \geq i+k} \widetilde{\mathcal{H}}_{Y_i,Y_j} \subseteq \bigoplus_{i,j} \widetilde{\mathcal{H}}_{Y_i,Y_j}.$$

Note that the graded quotients are isomorphic to the abelian Lie $\underline{\mathbb{Q}_p}$-algebra

$$\mathrm{Fil}^k / \mathrm{Fil}^{k+1} \simeq \bigoplus_i \widetilde{\mathcal{H}}_{Y_i, Y_{i+k}}.$$

4.1.2. *Automorphisms.* Let $\mathbf{Aut}(\tilde{Y})$ be functor of automorphisms of $\tilde{Y}$, then there is a monomorphism

$$\begin{aligned} \alpha : \mathbf{Aut}(\tilde{Y}) &\to \left(\widetilde{\mathcal{H}}_Y\right)^{\oplus 2} \\ \gamma &\mapsto (\gamma, \gamma^{-1}) \end{aligned}$$

and the image consists of those pairs (γ, γ') such that $\gamma \circ \gamma' = 1 = \gamma' \circ \gamma$. It follows from this that α is representable by closed immersions and therefore $\mathbf{Aut}(\tilde{Y})$ is a formal scheme. If we intersect with the subscheme $\underline{\mathrm{Hom}}(Y, Y)^{\oplus 2} = T_p \mathcal{H}_Y^{\oplus 2}$ we recover the group scheme $\mathbf{Aut}(Y)$. The projection to the diagonal map

$$\mathbf{Aut}(\tilde{Y}) \to \underline{\mathbf{Aut}(\tilde{Y})(\overline{\mathbb{F}}_p)},$$

has connected kernel $\mathbf{Aut}(\tilde{Y})^\circ$. This implies that we have a semi-direct product decomposition

$$\mathbf{Aut}(\tilde{Y}) \simeq \mathbf{Aut}(\tilde{Y})^\circ \rtimes \underline{\mathbf{Aut}(\tilde{Y})(\overline{\mathbb{F}}_p)}.$$

4.1.3. Our matrix description implies that all elements γ of $\mathbf{Aut}(\tilde{Y})^\circ$ are unipotent automorphisms of $\tilde{Y}$. This means that the logarithm map

$$\begin{aligned} L : \mathbf{Aut}(\tilde{Y})^\circ &\to \widetilde{\mathcal{H}}_Y^\circ \\ X &\mapsto \sum_{i=1}^{\infty} (-1)^{i+1} \tfrac{(X-1)^i}{i} \end{aligned}$$

and the exponential map

$$\begin{aligned} E : \widetilde{\mathcal{H}}_Y^\circ &\to \mathbf{Aut}(\tilde{Y})^\circ \\ X &\mapsto \sum_{i=0}^{\infty} \tfrac{X^i}{i!} \end{aligned}$$

are well defined. We know that $E \circ L = 1$ and $L \circ E = 1$ because we can check this in $\mathbb{Q}[[X]]$ by universality and there the result follows from complex analysis.

Remark 4.1.4. In particular, $\mathbf{Aut}(\tilde{Y})^\circ$ is isomorphic as a functor to $\widetilde{\mathcal{H}}_Y^\circ$, and so it is representable by a formal scheme. In fact, it is representable by a filtered colimit of spectra qrsp rings, because we can write it as

$$\widetilde{\mathcal{H}}_Y^\circ = \varinjlim_n \tfrac{1}{p^n} T_p \mathcal{H}_Y^\circ,$$

and the p-adic Tate module of a p-divisible group is qrsp. This implies that $\mathbf{Aut}(\tilde{Y})$ is also representable by a formal scheme which representable by a filtered colimit of spectra of qrsp rings.

The Baker–Campbell–Hausdorff formula gives an expression for the group structure on $\mathbf{Aut}(\tilde{Y})^\circ$ in terms of the Lie bracket on $\widetilde{\mathcal{H}}_Y^\circ$. In fact if V is a (possibly infinite dimensional) vector space over a field of characteristic zero and X, Y are two nilpotent endomorphisms of V, then the BCH formula expresses $\mathrm{Exp}(X)\,\mathrm{Exp}(Y)$ in terms of X and Y and the Lie bracket. This is explained in Chapter XVI of [37].

It follows from this that the filtration $\mathrm{Fil}^\bullet$ induces a filtration $\mathrm{Fil}^\bullet_{\mathbf{Aut}} = E(\mathrm{Fil}^\bullet)$ of $\mathbf{Aut}(\tilde{Y})^\circ$ by normal subgroups, with graded quotients isomorphic to

$$\bigoplus_i \widetilde{\mathcal{H}}_{Y_i,Y_{i+k}}.$$

Indeed, we can identify the graded pieces of the two filtrations via the exponential map, and the Baker–Campbell–Hausdorff formula tells us that the exponential map of an abelian Lie algebra is an isomorphism of groups.

4.2. **Dieudonné–Lie algebras.** This section is an interlude on Dieudonné modules endowed with a Lie bracket. Recall that $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$, let $\breve{\mathbb{Q}}_p = \breve{\mathbb{Z}}_p[1/p]$ and write φ for the Frobenius on both $\breve{\mathbb{Z}}_p$ and $\breve{\mathbb{Q}}_p$.

Definition 4.2.1. A *Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra* is a triple $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$ where $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+})$ is a Dieudonné module (see Definition 2.2.1) and

$$[-,-] : \mathfrak{a}^+ \times \mathfrak{a}^+ \to \mathfrak{a}^+$$

is a Lie bracket such that the following diagram commutes

$$\begin{tikzcd}
\varphi^*\mathfrak{a}^+[\frac{1}{p}] \times \varphi^*\mathfrak{a}^+[\frac{1}{p}] \arrow[r, "\varphi_{\mathfrak{a}^+} \times \varphi_{\mathfrak{a}^+}"] \arrow[d, "\varphi^*[-,-]"] & \mathfrak{a}^+[\frac{1}{p}] \times \mathfrak{a}^+[\frac{1}{p}] \arrow[d, "{[-,-]}"] \\
\varphi^*\mathfrak{a}^+[\frac{1}{p}] \arrow[r, "\varphi_{\mathfrak{a}^+}"] & \mathfrak{a}^+[\frac{1}{p}].
\end{tikzcd}$$

A morphism of Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras is a $\breve{\mathbb{Z}}_p$-linear map $f : \mathfrak{a}^+ \to \mathfrak{b}^+$ that respects the Lie brackets and induces a homomorphism of Dieudonné modules. We refer to objects of the isogeny category of Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras as *Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebras.* We write $(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$ for the Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra associated to $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$.

Example 4.2.2. The first example of a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra is the Dieudonné-module of the internal hom p-divisible group $\mathcal{H}_Y$ attached to a p-divisible group Y over $\overline{\mathbb{F}}_p$, denoted by $\mathbb{D}(\mathcal{H}_Y)$. Indeed, the Lie-bracket coming from the commutator bracket on

$$T_p\mathcal{H}_Y = \underline{\mathrm{Hom}}(Y,Y),$$

induces an φ-equivariant Lie bracket on $\mathfrak{a}$ by Corollary 1.2.5 of [36]. The Lie bracket on $\underline{\mathrm{Hom}}(Y,Y)$ clearly sends the identity component $\underline{\mathrm{Hom}}(Y,Y)^\circ$ to itself and so our second, and more fundamental, example of a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra is $\mathbb{D}(\mathcal{H}_Y^\circ)$.

4.2.3. We say that a Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra $(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$ is nilpotent if the underlying Lie algebra $(\mathfrak{a}, [-,-])$ is nilpotent, which is the case in our fundamental example. Note that it follows from the definition that the lower central series

$$\mathfrak{a} \supseteq [\mathfrak{a},\mathfrak{a}] \supseteq [\mathfrak{a},[\mathfrak{a},\mathfrak{a}]] \supseteq \cdots$$

is a filtration by F-stable Lie subalgebras, which we will write as

$$\mathfrak{a} = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \cdots \mathfrak{a}_{n(\mathfrak{a})} = 0.$$

The integer $n(\mathfrak{a})$ is called the *nilpotency class* of $\mathfrak{a}$. We will write

$$\mathfrak{a}_i^+ = \mathfrak{a}^+ \cap \mathfrak{a}_i,$$

which gives us a filtration of $\mathfrak{a}^+$ by Dieudonné–Lie $\breve{\mathbb{Z}}_p$-subalgebras which satisfy

$$[\mathfrak{a}^+, \mathfrak{a}_i^+] \subseteq \mathfrak{a}_{i+1}^+.$$

Moreover the graded quotients $\mathfrak{a}_i^+/\mathfrak{a}_{i+j}^+$ again have the structure of Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras.

Lemma 4.2.4. *Let $(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$ be a Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra where all the slopes are strictly smaller than 0. Let μ_1 be the smallest slope of $\mathfrak{a}$ and let $\mathfrak{b} \subseteq \mathfrak{a}$ be the maximal F-stable $\breve{\mathbb{Q}}_p$-subspace that is isoclinic of that slope. Then $\mathfrak{b}$ is contained in the centre of $\mathfrak{a}$.*

Proof. There are no nonzero F-equivariant maps

$$\mathfrak{b} \otimes \mathfrak{a} \to \mathfrak{a}$$

because all the slopes of $\mathfrak{b} \otimes \mathfrak{a}$ are strictly smaller than the slopes of $\mathfrak{a}$ since $\mathfrak{a}$ has strictly negative slopes. Hence the restriction of the Lie-bracket to $\mathfrak{b} \times \mathfrak{a}$ is trivial. $\square$

4.2.5. *Dieudonné–theory and bilinear maps.* Let $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$ be a Dieudonné–Lie algebra and let $\mathbb{X}(\mathfrak{a}^+)$ be the unique p-divisible group over $\overline{\mathbb{F}}_p$ with (covariant) Dieudonné module $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+})$. We would like to equip its universal cover and its Tate-module with a bilinear map, coming from the Lie bracket on $\mathfrak{a}^+$. For this we record a result that tells us how Dieudonné-theory for universal covers of p-divisible groups interacts with $\mathbb{Q}_p$-bilinear maps. The analogous result for bihomomorphisms of finite flat group schemes is Corollary 1.2.5 of [36]; we've decided to include the below proof for the benefit of the reader because it is quite short.

Lemma 4.2.6. *Let Y_1, Y_2, Y_3 be p-divisible groups over $\overline{\mathbb{F}}_p$, then there is a functorial and $\mathbb{Q}_p$-linear bijection between the space of $\breve{\mathbb{Q}}_p$-bilinear maps*

$$g : \mathbb{D}(Y_1)[\tfrac{1}{p}] \times \mathbb{D}(Y_2)[\tfrac{1}{p}] \to \mathbb{D}(Y_3)[\tfrac{1}{p}]$$

that satisfy $(g(\varphi_{Y_1}x, \varphi_{Y_2}y) = \varphi_{Y_3}(g(x,y))$ and the space of bilinear maps

$$f : \tilde{Y}_1 \times \tilde{Y}_2 \to \tilde{Y}_3.$$

Moreover if $Y_1 = Y_2 = Y_3$ then f satisfies the Jacobi identity if and only if g does.

Proof. Recall that the internal hom p-divisible group $\mathcal{H}_{Y_2,Y_3}$ satisfies

$$\widetilde{\mathcal{H}}_{Y_2,Y_3} = \underline{\mathrm{Hom}}(Y_2, Y_3)[\tfrac{1}{p}] = \underline{\mathrm{Hom}}(\tilde{Y}_2, \tilde{Y}_3).$$

It follows from the usual tensor-hom adjunction for $\mathbb{Q}_p$-vector spaces that bilinear maps

$$\tilde{Y}_1 \times \tilde{Y}_2 \to \tilde{Y}_3$$

are in bijection with homomorphisms

$$\tilde{Y}_1 \to \widetilde{\mathcal{H}}_{Y_2,Y_3}.$$

It also follows from the tensor-hom adjunction in the category of F-isocrystals that $\breve{\mathbb{Q}}_p$-bilinear maps

$$\mathbb{D}(Y_1)[\tfrac{1}{p}] \times \mathbb{D}(Y_2)[\tfrac{1}{p}] \to \mathbb{D}(Y_3)[\tfrac{1}{p}]$$

that 'commute with the Frobenius' as above are in bijection with morphisms of F-isocrystals

$$\mathbb{D}(Y_1)[\tfrac{1}{p}] \to \underline{\mathrm{Hom}}(\mathbb{D}(Y_2)[\tfrac{1}{p}], \mathbb{D}(Y_3)[\tfrac{1}{p}]),$$

where $\underline{\mathrm{Hom}}$ denotes the internal hom in the category of F-isocrystals. Because the slope of $\mathbb{D}(Y_1)[\frac{1}{p}]$ is bounded above by 0, these are also in bijection with morphisms of F-crystals

$$\mathbb{D}(Y_1)[\tfrac{1}{p}] \to \underline{\mathrm{Hom}}(\mathbb{D}(Y_2)[\tfrac{1}{p}], \mathbb{D}(Y_3)[\tfrac{1}{p}])^{\leq 0},$$

where $(-)^{\leq 0}$ denotes taking the slope ≤ 0 subspace. Lemma 4.1.7 of [9] tells us that there is an isomorphism

$$\mathbb{D}(\mathcal{H}_{Y_2,Y_3})[\tfrac{1}{p}] = \underline{\mathrm{Hom}}(\mathbb{D}(Y_2)[\tfrac{1}{p}], \mathbb{D}(Y_3)[\tfrac{1}{p}])^{\leq 0}$$

and Dieudonné-theory over $\overline{\mathbb{F}}_p$ tells us that homomorphisms

$$\tilde{Y}_1 \to \widetilde{\mathcal{H}}_{Y_2,Y_3}$$

are in bijection with morphisms of F-isocrystals

$$\mathbb{D}(Y_1)[\tfrac{1}{p}] \to \mathbb{D}(\mathcal{H}_{Y_2,Y_3})[\tfrac{1}{p}],$$

which is what we wanted to prove. We can similarly prove a statement about trilinear maps and use this to deduce that the Jacobi identity, which is just the vanishing of a certain trilinear map, is the same condition on both sides. $\square$

Remark 4.2.7. The map from bilinear maps of rational Dieudonné-modules to bilinear maps of universal covers of p-divisible groups can be described explicitly on R-points for qrsp R using the isomorphism

$$\tilde{Y}_i(R) = \left(B^+_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Q}}_p} \mathbb{D}(Y_i)[\tfrac{1}{p}]\right)^{\varphi=1}$$

of Lemma 2.3.7. Indeed, a $\breve{\mathbb{Q}}_p$-bilinear map

$$\mathbb{D}(Y_1)[\tfrac{1}{p}] \times \mathbb{D}(Y_2)[\tfrac{1}{p}] \to \mathbb{D}(Y_3)[\tfrac{1}{p}]$$

that ‘commutes with the Frobenius’ as above induces a $B^+_{\mathrm{cris}}(R)$-bilinear map

$$B^+_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Q}}_p} \mathbb{D}(Y_1)[\tfrac{1}{p}] \times B^+_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Q}}_p} \mathbb{D}(Y_2)[\tfrac{1}{p}] \to B^+_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Q}}_p} \mathbb{D}(Y_3)[\tfrac{1}{p}]$$

that induces a $\mathbb{Q}_p$-bilinear map on the $\varphi = 1$ subspaces.

Corollary 4.2.8. *Suppose that we are given a $\breve{\mathbb{Z}}_p$-bilinear map*

$$g^+ : \mathbb{D}(Y_1) \times \mathbb{D}(Y_2) \to \mathbb{D}(Y_3)$$

satisfying $g^+(\varphi_{Y_1}(x), \varphi_{Y_2}(y)) = \varphi_{Y_3}(g^+(x,y))$. The induced map $f : \tilde{Y}_1 \times \tilde{Y}_2 \to \tilde{Y}_3$ restricts to a $\mathbb{Z}_p$-bilinear Lie bracket

$$f^+ : T_pY_1 \times T_pY_2 \to T_pY_3.$$

Proof. It suffices to show this on R-valued points for semiperfect R and in fact by Yoneda it suffices to check it in the universal case that R is the ring underlying $T_pY_1 \times T_pY_2$. But this R is qrsp and so we can use Lemma 2.3.7 which tells us that

$$T_pY_i(R) = \left(\mathrm{A}_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Q}}_p} \mathbb{D}(Y_i)\right)^{\varphi=1},$$

and we see that f^+ sends $T_pY_1(R) \times T_pY_2(R)$ to $T_pY_3(R)$ because g^+ sends $\mathbb{D}(Y_1) \times \mathbb{D}(Y_2)$ to $\mathbb{D}(Y_3)$. $\square$

Definition 4.2.9. Let $n(\mathfrak{a})$ be the nilpotency class of the Lie algebra $\mathfrak{a}$, i.e., the length of its lower central series. We define $\tilde{\Pi}(\mathfrak{a})$ to be the formal group obtained by endowing $\tilde{\mathbb{X}}(\mathfrak{a})$ with the group structure given by the Baker–Campbell–Hausdorff formula. If $p > n(\mathfrak{a})$, then the Baker–Campbell–Hausdorff formula has coefficients in $\mathbb{Z}_{(p)}$ by Theorem 1 of [38]. Therefore, the group structure on $\tilde{\mathbb{X}}(\mathfrak{a})$ given by the BCH formula restricts to a group structure on $T_p\mathbb{X}(\mathfrak{a}^+)$, and we let $\Pi(\mathfrak{a}^+)$ denote this group scheme.

Hypothesis 4.2.10. From now on we will assume that $p > n(\mathfrak{a})$.

Remark 4.2.11. Let Y be a p-divisible group with internal hom p-divisible group $\mathcal{H}_Y$ and consider the Dieudonné–Lie algebra $\mathbb{D}(\mathcal{H}_Y^\circ)$ from Example 4.2.2. The analysis done in Section 4.1 tells us that $\mathfrak{a}$ is a nilpotent Lie algebra and gives us an identification

$$\Pi(\mathfrak{a}^+) = \mathbf{Aut}(Y)^\circ$$
$$\tilde{\Pi}(\mathfrak{a}) = \mathbf{Aut}(\tilde{Y})^\circ.$$

Lemma 4.2.12. *The group scheme $\Pi(\mathfrak{a}^+)$ is quasisyntomic and semiperfect.*

Proof. It follows from Proposition 4.1.2.(4) of [9] that for any connected p-divisible group $Y/\overline{\mathbb{F}}_p$ the Tate-module T_pY is representable by a scheme of the form

$$\overline{\mathbb{F}}_p[[x_1^{1/p^\infty}, \cdots, x_n^{1/p^\infty}]]/(x_1, \cdots, x_n),$$

which is qrsp. Since $\Pi(\mathfrak{a}^+) \simeq T_p\mathbb{X}(\mathfrak{a}^+)$ as schemes, we are done. $\square$

Let us record the following representability result.

Proposition 4.2.13. *The quotient of $\tilde{\Pi}(\mathfrak{a})$ by the subgroup $\Pi(\mathfrak{a}^+)$ is representable by a formally smooth formal scheme of dimension equal to $\dim \mathbb{X}(\mathfrak{a}^+)$.*

Proof. What we have proved above implies that

$$\frac{\tilde{\Pi}(\mathfrak{a}_i)/\Pi(\mathfrak{a}_i^+)}{\tilde{\Pi}(\mathfrak{a}_{i+1})/\Pi(\mathfrak{a}_{i+1}^+)} = \frac{\tilde{\mathbb{X}}\left(\frac{\mathfrak{a}_i}{\mathfrak{a}_{i+1}}\right)}{T_p\mathbb{X}\left(\frac{\mathfrak{a}_i^+}{\mathfrak{a}_{i+1}^+}\right)} = \mathbb{X}\left(\frac{\mathfrak{a}_i^+}{\mathfrak{a}_{i+1}^+}\right),$$

which is a p-divisible group and hence representable by a formally smooth formal scheme. Now consider the tower

$$\frac{\tilde{\Pi}(\mathfrak{a})}{\Pi(\mathfrak{a}^+)} \to \frac{\tilde{\Pi}(\mathfrak{a})/\Pi(\mathfrak{a}^+)}{\tilde{\Pi}(\mathfrak{a}_{n(\mathfrak{a})-1})/\Pi(\mathfrak{a}_{n(\mathfrak{a})-1}^+)} \to \cdots \to \frac{\tilde{\Pi}(\mathfrak{a})/\Pi(\mathfrak{a}^+)}{\tilde{\Pi}(\mathfrak{a}_1)/\Pi(\mathfrak{a}_1^+)}.$$

The representability of $\frac{\tilde{\Pi}(\mathfrak{a})}{\Pi(\mathfrak{a}^+)}$ follows by induction because the k-th object of the tower is a torsor under a p-divisible group over the $(k-1)$-th object, and torsors under p-divisible groups are formally smooth and representable by formal schemes. $\square$

Definition 4.2.14. We denote by $Z(\mathfrak{a}^+)$ the formally smooth formal scheme $\frac{\tilde{\Pi}(\mathfrak{a})}{\Pi(\mathfrak{a}^+)}$.

4.3. **Automorphism groups for Shimura varieties of Hodge type.** Let Y be a p-divisible group over $\overline{\mathbb{F}}_p$ and fix an isomorphism $\mathbb{D}(Y)[1/p] \simeq V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p$, where V is a vector space over $\mathbb{Q}_p$. In order to generalise Section 4.1 to Shimura varieties of Hodge type, we let $b \in \mathrm{GL}_V(\breve{\mathbb{Q}}_p)$ be the Frobenius of the F-isocrystal $\mathbb{D}(Y)[\frac{1}{p}]$. Then the internal hom F-isocrystal

$$\underline{\mathrm{Hom}}(\mathbb{D}(Y)[\tfrac{1}{p}], \mathbb{D}(Y)[\tfrac{1}{p}])$$

is isomorphic to the F-isocrystal

$$(\mathfrak{gl}_V \otimes \breve{\mathbb{Q}}_p, \mathrm{Ad}\, \sigma\, b).$$

If we are given a reductive group $G \subseteq \mathrm{GL}_V$ such that $b \in G(\breve{\mathbb{Q}}_p)$, then we get an inclusion of F-isocrystals

$$(\mathfrak{g} \otimes \breve{\mathbb{Q}}_p, \mathrm{Ad}\, \sigma\, b) \subseteq (\mathfrak{gl}_V \otimes \breve{\mathbb{Q}}_p, \mathrm{Ad}\, \sigma\, b).$$

The slope filtration of this F-isocrystal is described in Section 3 of [47]: the slope t part is given by

$$\bigoplus_{\substack{\alpha_0 \in \Phi_0^+ \\ \langle \alpha_0, \nu_b \rangle = t}} \mathfrak{u}_{\alpha_0},$$

where α_0 runs over the relative roots of $G_{\breve{\mathbb{Q}}_p}$ with respect to a maximal $\breve{\mathbb{Q}}_p$ split torus S defined over $\mathbb{Q}_p$ that is contained in a Borel subgroup B with respect to which the Newton cocharacter ν_b of b is dominant (see [54, Section 1.1.2] for the definition of the Newton cocharacter). We see that the slope ≤ 0 part corresponds precisely to Lie algebra of the standard parabolic subgroup $P_b = P_{\nu_b}$ associated to ν_b and that the slope 0 part corresponds to the Lie algebra of the Levi M_b.

Taking non-positive slope parts we get an F-stable sub-isocrystal

$$\mathrm{Lie}\, P_{\nu_b} \subseteq \mathbb{D}(\mathcal{H}_Y)[\tfrac{1}{p}],$$

and intersecting with $\mathbb{D}(\mathcal{H}_Y)$ we get a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra which we will write as $\mathbb{D}(\mathcal{H}_Y^G)$ for a p-divisible group $\mathcal{H}_Y^G \subset \mathcal{H}_Y$. Note that by construction and Corollary 4.2.8

$$T_p \mathcal{H}_Y^G \subset T_p \mathcal{H}_Y = \underline{\mathrm{Hom}}(Y, Y)$$

is stable under the commutator bracket.

Corollary 4.3.1. *The dimension of $\mathcal{H}_Y^G$ is equal to $\langle 2\rho, \nu_b \rangle$.*

Proof. This is explained at the end of the proof of Proposition 3.1.4 of [47]. $\square$

Thanks to Lemma 4.2.6, we have that

$$\widetilde{\mathcal{H}}_Y^G \subseteq \widetilde{\mathcal{H}}_Y = \underline{\mathrm{Hom}}(Y, Y)[\tfrac{1}{p}]$$

is closed under the Lie bracket (the commutator) of $\underline{\mathrm{Hom}}(Y, Y)[\frac{1}{p}]$. The Dieudonné module

$$\mathbb{D}(\mathcal{H}_Y^{G,\circ})[\tfrac{1}{p}]$$

is stable under the Lie bracket and consists precisely of the strictly negative slope part of the isocrystal $\mathrm{Lie}\, P_{\nu_b}$, which can be identified with

$$\mathrm{Lie}\, U_{\nu_b} \subseteq \mathrm{Lie}\, P_{\nu_b},$$

where U_{ν_b} is the unipotent radical of P_{ν_b}.

4.3.2. Write $V^{\otimes}$ for the direct sum of $V^{\otimes n} \otimes (V^*)^{\otimes m}$ for all pairs of integers $m \geq 0, n \geq 0$. We will also use this notation later for modules over commutative rings and modules over sheaves of rings.

As in [54, Sec. 1.3.4], we can choose tensors $\{s_\alpha\} \subset V^{\otimes}$ such that G is their pointwise stabiliser in GL_V. If we identify the Lie algebra $\mathfrak{gl}_V$ of GL_V with the vector space of endomorphisms of V, then by Lemma 5.3.3 of [39] $\mathfrak{g} \subset \mathfrak{gl}_V$ consists of those endomorphisms g satisfying $g^{\otimes} s_\alpha = 0$, let us call such endomorphisms *tensor annihilating endomorphisms*. It follows that

$$\mathbb{D}(\mathcal{H}_Y^{G,\circ})[\tfrac{1}{p}] \subseteq \mathbb{D}(\mathcal{H}_Y)[\tfrac{1}{p}]$$

is the subspace of $\mathbb{D}(\mathcal{H}_Y)[\frac{1}{p}]$ of tensor annihilating endomorphisms. Therefore by Remark 4.2.7 it follows that for qrsp $\overline{\mathbb{F}}_p$-algebras R we have that

$$\widetilde{H}_Y^G(R) \subset \widetilde{H}_Y(R) = \mathrm{Hom}(Y_R, Y_R)[1/p]$$

consists of the endomorphisms $\tilde{Y}_R \to \tilde{Y}_R$ such that the induced endomorphism

$$\mathbb{D}(g) : \mathbb{D}(Y)[1/p] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R) \to \mathbb{D}(Y)[1/p] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R)$$

satisfies $g^{\otimes}(s_\alpha \otimes 1) = 0$ for all α. It follows that

$$T_p\mathcal{H}_Y^G(R) \subset T_p\mathcal{H}_Y(R) = \mathrm{Hom}(Y_R, Y_R)$$

consists precisely of the endomorphisms $g : Y_R \to Y_R$ such that the induced endomorphism

$$\mathbb{D}(g) : \mathbb{D}(Y)[1/p] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R) \to \mathbb{D}(Y)[1/p] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R)$$

satisfies $g^{\otimes}(s_\alpha \otimes 1) = 0$ for all α. In both cases we will use the term *tensor annihilating endomorphisms* to denote such endomorphisms of Y_R or of $\tilde{Y}_R$.

4.3.3. There is also the notion of *tensor preserving automorphism* of $\tilde{Y}_R$ which is an automorphism g of $\tilde{Y}_R$ such that the induced automorphism

$$\mathbb{D}(g) : \mathbb{D}(Y)[1/p] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R) \to \mathbb{D}(Y)[1/p] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R)$$

satisfies $g^{\otimes}(s_\alpha \otimes 1) = (s_\alpha \otimes 1)$ for all α.

Lemma 4.3.4. *There is a closed subgroup*

$$\mathbf{Aut}_G(\tilde{Y}) \subseteq \mathbf{Aut}(\tilde{Y})$$

such that on qrsp $\overline{\mathbb{F}}_p$-algebras R the subgroup

$$\mathbf{Aut}_G(\tilde{Y})(R) \subseteq \mathbf{Aut}(\tilde{Y})(R)$$

consists precisely of the tensor preserving automorphisms. Moreover, there is an isomorphism of formal groups

$$\mathbf{Aut}_G(\tilde{Y}) \simeq \mathbf{Aut}_G(\tilde{Y})^\circ \rtimes \underline{J_b(\mathbb{Q}_p)},$$

where $\mathbf{Aut}_G(\tilde{Y})^\circ$ is the intersection of $\mathbf{Aut}(\tilde{Y})^\circ$ with $\mathbf{Aut}_G(\tilde{Y})$ inside $\mathbf{Aut}(\tilde{Y})$ and $J_b(\mathbb{Q}_p) \subset G(\breve{\mathbb{Q}}_p)$ is the twisted centraliser of b.

Proof. It is clear from the definition that the exponential of a nilpotent tensor annihilating endomorphism of $\tilde{Y}_R$ is a unipotent tensor preserving automorphism of $\tilde{Y}_R$. Conversely the logarithm of a unipotent tensor preserving automorphism of $\tilde{Y}_R$ is a nilpotent tensor annihilating endomorphism of $\tilde{Y}_R$. Since $\mathbf{Aut}(\tilde{Y})^\circ \subset \mathbf{Aut}(\tilde{Y})$ is precisely the subgroup of unipotent automorphisms, the exponential map defines an isomorphism of functors

$$\mathcal{H}_Y^{G,\circ} \simeq \mathbf{Aut}_G(\tilde{Y})^\circ,$$

and thus $\mathbf{Aut}_G(\tilde{Y})^\circ \subset \mathbf{Aut}(\tilde{Y})^\circ$ is representable by closed immersions. We have seen that there is a semi-direct product decomposition

$$\mathbf{Aut}(\tilde{Y}) \simeq \mathbf{Aut}(\tilde{Y})^\circ \rtimes \underline{\mathbf{Aut}(\tilde{Y})(\overline{\mathbb{F}}_p)}.$$

There is a closed subgroup of the locally profinite group $\underline{\mathbf{Aut}(\tilde{Y})(\overline{\mathbb{F}}_p)}$ consisting of those automorphisms of $\tilde{Y}$ that are tensor preserving. By Dieudonné-theory we can identify this group with the group of tensor preserving automorphisms of the F-isocrystal $\mathbb{D}(Y)[1/p]$; this group is precisely the twisted centraliser $J_b(\mathbb{Q}_p) \subset G(\breve{\mathbb{Q}}_p)$ of b.

The action of $J_b(\mathbb{Q}_p) \subset G(\breve{\mathbb{Q}}_p)$ on $\operatorname{Lie} G$ stabilises $\operatorname{Lie} U_{\nu_b}$ since $J_b(\mathbb{Q}_p)$ is contained in the centraliser of ν_b inside $G(\breve{\mathbb{Q}}_p)$. Therefore we get a closed subgroup

$$\mathbf{Aut}_G(\tilde{Y})^\circ \rtimes \underline{J_b(\mathbb{Q}_p)} \subset \mathbf{Aut}(\tilde{Y})^\circ \rtimes \underline{\mathbf{Aut}_G(\tilde{Y})(\overline{\mathbb{F}}_p)},$$

whose R-points for qrsp R give the group of tensor preserving automorphisms of $\tilde{Y}_R$. $\square$

Our construction does *not* agree with definition 2.3.1 of [47], which defines $\mathbf{Aut}_G(\tilde{Y})$ as the intersection of

$$\left(\widetilde{\mathcal{H}}_Y^G \times \widetilde{\mathcal{H}}_Y^G\right) \cap \mathbf{Aut}(\tilde{Y}).$$

By the discussion above, the R-points of this functor are given by automorphisms g of $\tilde{Y}_R$ such that the induced automorphism

$$\mathbb{D}(g) : \mathbb{D}(Y)[1/p] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R) \to \mathbb{D}(Y)[1/p] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R) \tag{4.3.1}$$

satisfies $g^\otimes(s_\alpha \otimes 1) = 0$ for all α. Note that an automorphism g of $\tilde{Y}_R$ induces an automorphism of $\mathbb{D}(Y)[1/p] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R)$. Thus if g is an R-point of this intersection then $g^\otimes(s_\alpha \otimes 1)$ cannot be zero unless $s_\alpha \otimes 1 = 0$ which implies that $s_\alpha = 0$. Therefore the functor (4.3.1) is empty unless $G = \mathrm{GL}_V$.

Fortunately, the rest of [47] only uses the characterisation of $\mathbf{Aut}_G(\tilde{Y})$ of Lemma 4.3.4. For instance Proposition 3.2.4 of [47] is correct as stated. Therefore the rest of [47] is not affected.

We end this section by defining

$$\mathbf{Aut}_G(Y) = \mathbf{Aut}_G(\tilde{Y}) \cap \mathbf{Aut}(Y),$$

and noting that the image of its identity component $\mathbf{Aut}_G(Y)^\circ$ under the logarithm map is given by $T_p\mathcal{H}_Y^{G,\circ}$ if $p > n(\operatorname{Lie} U_{\nu_b})$. In particular under this hypothesis we can identify the group schemes

$$\Pi(\mathbb{D}(\mathcal{H}_Y^\circ)) \simeq \mathbf{Aut}_G(Y)^\circ.$$

4.4. Let $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$ be a nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra with associated Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra $(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$. Let $\mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}})$ be the automorphism group of the underlying F-isocrystal, considered as an algebraic group over $\mathbb{Q}_p$ as in §1.11 of [67]. Then the Lie algebra of $\mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}})$ can be identified with the endomorphism ring of the F-isocrystal, equipped with the commutator bracket.

There is a closed subgroup

$$\mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-]) \subseteq \mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}})$$

consisting of those automorphisms preserving the Lie bracket.

Definition 4.4.1. Let Q be an algebraic group over $\mathbb{Q}_p$ equipped with a group homomorphism

$$Q \to \mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-]).$$

We call such a group homomorphism (or action) *strongly non-trivial* if the induced linear representation of Q on $\mathfrak{a}$ has no non-trivial subquotients.

In the situation of Definition 4.4.1, the elements of $Q(\mathbb{Q}_p)$ act via $F = \varphi_{\mathfrak{a}}$-equivariant $\breve{\mathbb{Q}}_p$-linear Lie algebra automorphisms on $(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$ and thus via Lie $\underline{\mathbb{Q}_p}$-algebra automorphisms on $\tilde{\mathbb{X}}(\mathfrak{a})$ by functoriality. There is a compact open subgroup $\Gamma \subseteq Q(\mathbb{Q}_p)$ preserving $\mathfrak{a}^+$. By construction, the action of Γ on $\tilde{\mathbb{X}}(\mathfrak{a})$ preserves $T_p\mathbb{X}(\mathfrak{a}^+)$, therefore there is an induced action of Γ on the p-divisible group $\mathbb{X}(\mathfrak{a}^+)$.

Since all these actions preserve the Lie-bracket, there are induced actions on the unipotent groups $\Pi(\mathfrak{a}^+)$ and $\tilde{\Pi}(\mathfrak{a})$. Therefore there is an induced action of Γ on the formally smooth formal scheme $Z(\mathfrak{a}^+)$.

Example 4.4.2. If $\mathfrak{a} = \operatorname{Lie} U_{\nu_b}$ for some element $b \in G(\breve{\mathbb{Q}}_p)$ admissible with respect to some Shimura datum on G, then the algebraic group J_b has a natural and strongly non-trivial action on $\mathfrak{a}$. Moreover, by Proposition 5.5.1 of [39] the restriction of this action to a maximal torus $T \subseteq J_b$ is still strongly non-trivial.

The map $Q \to \mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}})$ induces a Lie algebra homomorphism

$$\operatorname{Lie} Q \to \operatorname{Lie} \mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-]) \to \operatorname{Lie} \mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}}) = \operatorname{End}(\mathfrak{a}, \varphi_{\mathfrak{a}})$$

and a routine computation shows that the Lie algebra of $\mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$ consist of all those homomorphism of F-isocrystals $g : \mathfrak{a} \to \mathfrak{a}$ such that for all $x, y \in \mathfrak{a}$ we have

$$[gx, gy] + [x, gy] + [gx, y] = 0. \tag{4.4.1}$$

Let $\operatorname{Lie} \Gamma$ be the Lie algebra of Γ in the sense of p-adic groups. Then there is a Lie-bracket stable $\mathbb{Z}_p$ lattice $\Lambda \subset \operatorname{Lie} \Gamma$ stabilising $\mathfrak{a}^+$, and we may assume that the exponential converges on Λ and takes values in Γ, after possibly shrinking Λ.

Now elements of Λ induce endomorphisms of the Dieudonné module $\mathfrak{a}^+$ satisfying (4.4.1), and by functoriality these will induce endomorphisms of $\tilde{\mathbb{X}}(\mathfrak{a})$ preserving $T_p\mathbb{X}(\mathfrak{a}^+)$.

Lemma 4.4.3 (Continuity)**.** *The action of the (abstract group) $\mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])(\mathbb{Q}_p)$ on $\tilde{\mathbb{X}}(\mathfrak{a})$ can be upgraded uniquely to an action of the locally profinite group scheme*

$$\underline{\mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])(\mathbb{Q}_p)}.$$

Proof. By the definition of the locally profinite topology on $\mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])(\mathbb{Q}_p)$ it suffices to show that for sufficiently small compact open subgroups U, the action of U on $\mathfrak{a}$ extends to an action of the profinite group scheme $\underline{U}$. In particular we may assume that U preserves $\mathfrak{a}^+$ and therefore its action on $\tilde{\mathbb{X}}(\mathfrak{a})$ preserves the Tate module $T_p\mathbb{X}(\mathfrak{a}^+)$ of $\mathbb{X}(\mathfrak{a}^+)$.

Since the action of U on $T_p\mathbb{X}(\mathfrak{a}^+)$ is by group homomorphisms, there are induced actions of U on the quotients

$$T_p\mathbb{X}(\mathfrak{a}^+)/p^n T_p\mathbb{X}(\mathfrak{a}^+) = \mathbb{X}(\mathfrak{a}^+)[p^n]$$

of $T_p\mathbb{X}(\mathfrak{a}^+)$.

Claim 4.4.4. *The action of U on these quotients factors through a finite quotient of U.*

Proof. We can check this after shrinking U and so we may assume that there is a Lie bracket stable lattice $\Lambda \subset \operatorname{Lie} \mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$ such that the exponential converges on Λ and has image $U \subset \mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])(\mathbb{Q}_p)$. There is an integer $m \gg 0$ and a commutative diagram

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d, "\mathrm{Exp}"] & p^m \operatorname{End}(\mathbb{X}) \arrow[d, "\mathrm{Exp}"] \\ U \arrow[r] & \operatorname{Aut}(\mathbb{X}). \end{tikzcd}$$

The action of $\operatorname{End}(\mathbb{X})$ on $\mathbb{X}[p^n]$ factors through $\operatorname{End}(\mathbb{X}[p^n])$ via a finite quotient, and so the action of $\operatorname{Aut}(\mathbb{X})$ on $\mathbb{X}[p^n]$ factors through a finite quotient. $\square$

It follows from the claim that the action of U induces a unique action of $\underline{U}$ on $\mathbb{X}(\mathfrak{a}^+)[p^n]$. Since

$$T_p\mathbb{X}(\mathfrak{a}^+) = \varprojlim_n \mathbb{X}(\mathfrak{a}^+)[p^n]$$

it follows that the action of U on $T_p\mathbb{X}(\mathfrak{a}^+)$ upgrades uniquely to an action of $\underline{U}$. Because the action of U commutes with multiplication by p it follows that the action of U on $\frac{1}{p^n}T_p\mathbb{X}(\mathfrak{a}^+)$ upgrades uniquely to an action of $\underline{U}$ and since

$$\tilde{\mathbb{X}}(\mathfrak{a}) = \varinjlim_n \tfrac{1}{p^n}T_p\mathbb{X}(\mathfrak{a}^+)$$

we deduce that the action of U on $\tilde{\mathbb{X}}(\mathfrak{a})$ upgrades uniquely to an action of $\underline{U}$. $\square$

5. Deformation theory of central leaves

5.1. **Introduction.** In this section we will recall the constructions of the canonical integral models of Shimura varieties of Hodge type at hyperspecial level from [51]. We then recall the definitions of central leaves $C_{G,[\![b]\!]}$ inside the special fibers $\mathrm{Sh}_{G,U}$ of these canonial integral models, and also the definition of Igusa varieties from [33, 34, 47]. The main goal of this section is to study the structure of the formal completions $C^{/x}_{G,[\![b]\!]}$ of central leaves at points $x \in C_{G,[\![b]\!]}(\overline{\mathbb{F}}_p)$. Using the Hodge embedding, these formal schemes admit closed immersions to deformation spaces $\mathbf{Def}(Y)$ of p-divisible groups Y over $\overline{\mathbb{F}}_p$ and in fact they admit closed immersions to central leaves $\mathbf{Def}_{\mathrm{sus}}(Y) \subseteq \mathbf{Def}(Y)$ inside these deformation spaces[2]. If the corresponding p-divisible group is completely slope divisible, then $\mathbf{Def}_{\mathrm{sus}}(Y)$ has the structure of a p-divisible *cascade* in the sense of [62]. For example if Y has two slopes then $\mathbf{Def}_{\mathrm{sus}}(Y)$ has the structure of a p-divisible formal group. As observed already in [70] (cf. [42]), one cannot expect that $C^{/x}_{G,[\![b]\!]} \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$ is a sub-cascade but only that it is a *shifted subcascade*. We will not make use of p-divisible cascades or shifted subcascades in this paper and instead show, using the work of Caraiani–Scholze and Kim, that the perfections of the formal schemes $C^{/x}_{G,[\![b]\!]}$ and $\mathbf{Def}_{\mathrm{sus}}(Y)$ canonically have a (non-commutative) group structure. These groups will be of the form $\tilde{\Pi}(\mathfrak{a})$ for a Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra $\mathfrak{a}$ as introduced in Section 4.

In Section 5.6 we will show that F-stable Lie subalgebras $\mathfrak{b} \subset \mathfrak{a}$ gives rise to formally smooth closed subschemes of $\mathbf{Def}_{\mathrm{sus}}(Y)$. These are precisely the *strongly Tate-linear* subspaces of Chai and Oort. We end by stating a conjecture on the monodromy group of the universal isocrystal over such subspaces.

5.2. **Integral models.** For a symplectic space (V, ψ) over $\mathbb{Q}$, we write $\mathcal{G}_V := \mathrm{GSp}(V, \psi)$ for the group of symplectic similitudes of V over $\mathbb{Q}$. It admits a Shimura datum $(\mathcal{G}_V, \mathcal{H}_V)$, where $\mathcal{H}_V$ is the union of the Siegel upper and lower half-spaces. Let (G, X) be a Shimura datum of Hodge type with reflex field E and let $(G, X) \to (\mathcal{G}_V, \mathcal{H}_V)$ be a Hodge embedding.

Let $K_p \subset G(\mathbb{Q}_p)$ be a hyperspecial subgroup. Then after possibly changing the Hodge embedding and the symplectic space V, we can find a $\mathbb{Z}_{(p)}$-lattice $V_{(p)}$ on which ψ is $\mathbb{Z}_{(p)}$-valued, such that K_p is the stabiliser in $G(\mathbb{Q}_p)$ on $V_p := V_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$, see Section 2.3.15 of [50]. Write $G_{\mathbb{Z}_{(p)}}$ for the Zariski closure of G in $GL(V_{\mathbb{Z}_{(p)}})$, then $G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ is a reductive integral model $\mathcal{G}$ of G.

For every sufficiently small compact open subgroup $U^p \subset G(\mathbb{A}_f^p)$, we can find $\mathcal{U}^p \subset \mathcal{G}_V(\mathbb{A}_f^p)$ such that the Hodge embedding induces a closed immersion (see Lemma 2.1.2 of [51])

$$\mathbf{Sh}_U(G, X) \to \mathbf{Sh}_{\mathcal{U}}(\mathcal{G}_V, \mathcal{H}_V) \otimes_{\mathbb{Q}} E$$

[2]The notation $\mathbf{Def}_{\mathrm{sus}}$ denotes the *sustained deformation space* in the sense of Chai–Oort, this will be explained in Section 5.5.

of (canonical models of) Shimura varieties of levels $U = U^pU_p$ and $\mathcal{U} = \mathcal{U}^p\mathcal{U}_p$, respectively. We let $\mathcal{S}_{\mathcal{U}}$ over $\mathbb{Z}_{(p)}$ be the moduli-theoretic integral model of $\mathbf{Sh}_{\mathcal{U}}(\mathcal{G}_V, \mathcal{H}_V)$; it is a moduli space of polarised abelian schemes (A, λ) up to prime-to-p isogeny with level $\mathcal{U}^p$-structure. Fix a prime $v|p$ of E and let

$$\mathscr{S}_U := \mathscr{S}_U(G, X) \to \mathcal{S}_{\mathcal{U}} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)}$$

be the normalisation of the Zariski closure of $\mathbf{Sh}_U(G, X)$ in $\mathcal{S}_{\mathcal{U}} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)}$. Then by the main result of [51] and [49], see Section 2 of [49], the scheme $\mathbf{Sh}_U(G, X)$ is smooth and in fact isomorphic to the canonical integral model of $\mathbf{Sh}_U(G, X)$. The main result of [76] tells us that

$$\mathscr{S}_U := \mathscr{S}_U(G, X) \to \mathcal{S}_{\mathcal{U}} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)}$$

is a closed immersion. Choose an algebraic closure $\overline{\mathbb{F}}_p$ of the residue field of $\mathcal{O}_{E,(v)}$ and let $\mathrm{Sh}_{G,U}$ be the base change to $\overline{\mathbb{F}}_p$ of $\mathscr{S}_U$. We will write $\mathcal{A}_{g,U_V}$ for the base change to $\overline{\mathbb{F}}_p$ of $\mathcal{S}_{\mathcal{U}}$, where $g = \frac{1}{2} \operatorname{Dim} V$ Then the pullback of the universal abelian variety over $\mathcal{S}_{\mathcal{U}}$ gives a family of g-dimensional abelian varieties A over $\mathrm{Sh}_{G,U}$ with associated p-divisible group $X = A[p^\infty]$.

5.2.1. *Tensors.* Recall the notation $V_{\mathbb{Z}_{(p)}}^{\otimes}$ from Section 4.3.2. By Lemma 1.3.2 of [51], the subgroup $G_{\mathbb{Z}_{(p)}} \subset GL(V_{\mathbb{Z}_{(p)}})$ is the stabiliser of a collection of tensors $s_\alpha \in V_{\mathbb{Z}_{(p)}}^{\otimes}$.

For $x \in \mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p)$, we write A_x for the abelian variety over $\overline{\mathbb{F}}_p$ corresponding to the image of $x \in \mathcal{A}_{g,U_V}$. Let $x \in \mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p)$ and let $\mathbb{D}_{\mathrm{contr},x}$ be the (contravariant!)[3] Dieudonné-module of $A_x[p^\infty]$. It is explained in Section 6.3 of [70] that there are canonical tensors $\{s_{\alpha,\mathrm{cris}}, x\}$ in

$$\mathbb{D}_{\mathrm{contr},x}^{\otimes},$$

that are invariant under the Frobenius on $\mathbb{D}_{\mathrm{contr},x}[1/p]$. It is moreover explained there that there is an isomorphism

$$\mathbb{D}_{\mathrm{contr},x} \simeq V_{(p)} \otimes_{\mathbb{Z}_{(p)}} \breve{\mathbb{Z}}_p$$

taking $s_{\alpha,\mathrm{cris},x}$ to $s_\alpha \otimes 1$. Under such an isomorphism, the Frobenius corresponds to an element $b_x \in G(\breve{\mathbb{Q}}_p)$, which is well defined up to σ-conjugacy by $\mathcal{G}(\breve{\mathbb{Z}}_p)$, where $\sigma : G(\breve{\mathbb{Q}}_p) \to G(\breve{\mathbb{Q}}_p)$ is induced by the Frobenius $\sigma : \breve{\mathbb{Q}}_p \to \breve{\mathbb{Q}}_p$ (which we call φ before). We will denote by $[\![b_x]\!]$ the $\mathcal{G}(\breve{\mathbb{Z}}_p)$-$\sigma$-conjugacy class of b_x and by $[b_x]$ the $G(\breve{\mathbb{Q}}_p)$-σ-conjugacy class of b_x.

5.3. **Central leaves.** It follows from Corollary 3.3.8 of [34] that for $b \in G(\breve{\mathbb{Q}}_p)$ there are (reduced) locally closed subschemes

$$C_{G,[\![b]\!]} \subset \mathrm{Sh}_{G,U,[b]} \subset \mathrm{Sh}_{G,U}$$

of $\mathrm{Sh}_{G,U}$ such that their $\overline{\mathbb{F}}_p$-points are given by

$$C_{G,[\![b]\!]}(\overline{\mathbb{F}}_p) = \{x \in \mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p) \ : \ [\![b_x]\!] = [\![b]\!]\}$$
$$\mathrm{Sh}_{G,U,[b]}(\overline{\mathbb{F}}_p) = \{x \in \mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p) \ : \ [b_x] = [b]\}.$$

The subscheme $\mathrm{Sh}_{G,U,[b]}$ is called the *Newton stratum* attached to $[b]$, and the subscheme $C_{G,[\![b]\!]} \subset \mathrm{Sh}_{G,U,[b]}$ is called the *central leaf* attached to $[\![b]\!]$. We note that the natural map $C_{G,[\![b]\!]} \to \mathrm{Sh}_{G,U,[b]}$ is a closed immersion by Corollary 3.3.8 of [34] and that the central leaf $C_{G,[\![b]\!]}$ is smooth and equidimensional by Corollary 5.3.1 of [48]. The following remark is Remark 2.1.4 of [40].

[3] As in [39], we use both covariant and contravariant Dieudonné theory in this paper. Since we mostly use the covariant theory, we will denote all our contravariant Dieudonné-modules with the subscript contr. The reason for this is that all the work on integral models of Shimura varieties uses the contravariant theory, while results about internal-hom p-divisible groups are best expressed in terms the covariant theory.

Remark 5.3.1. When $(G, X) = (\mathcal{G}_V, \mathcal{H}_V)$, then the $\mathcal{G}(\breve{\mathbb{Z}}_p)$-conjugacy class $[\![b_x]\!]$ captures precisely the isomorphism class of the polarised p-divisible group $(A_x[p^\infty], \lambda_x)$, where an isomorphism of polarised p-divisible groups $f : (Y, \mu) \to (Y', \mu')$ is an isomorphism $f : Y \to Y'$ such that $f^*\mu' = c\mu$ for some $c \in \mathbb{Z}_p^\times$. In particular, when $(G, X) = (\mathcal{G}_V, \mathcal{H}_V)$ our central leaves do *not* agree with those defined in [17], which are defined using isomorphisms $f : (Y, \mu) \to (Y', \mu')$ with $f^*\mu' = \mu$.

Fix a point $x \in \mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p)$ and write $Y = A_x[p^\infty]$ and write λ for the induced polarisation. We write $[\![b]\!] := [\![b_x]\!]$ for the $\mathcal{G}(\breve{\mathbb{Z}}_p)$-$\sigma$-conjugacy class of elements of $G(\breve{\mathbb{Q}}_p)$ associated to x. We will write $C_{(Y,\lambda)} \subset \mathcal{A}_{g,U_V}$ for the central leaf associated to the polarised p-divisible group (Y, λ).

Hypothesis 5.3.2. The p-divisible group Y is completely slope divisible.

Proposition 2.4.5 of [47] tells us that for every Newton stratum $\mathrm{Sh}_{G,U,[b]} \subseteq \mathrm{Sh}_{G,U}$ we can always find a central leaf $C_{G,[\![b']\!]} \subseteq \mathrm{Sh}_{G,U,[b]}$ such that $C_{G,[\![b']\!]} \subset C_{(Y',\lambda')}$ where $C_{(Y',\lambda')}$ corresponds to a completely slope divisible p-divisible group Y'. Thus this is not an unreasonable assumption.

As explained in [56, Sec 3.2.3], this implies that $X = A[p^\infty]$ over $C_{(Y,\lambda)}$ admits a slope filtration and we will denote the associated graded pieces for the slope filtration by X_i. We can then consider the Igusa towers

$$\mathrm{Ig}_{\mathrm{CS},\lambda} \to \mathrm{Ig}_{\mathrm{M},\lambda} \to C_{(Y,\lambda)},$$

where $\mathrm{Ig}_{\mathrm{M},\lambda} \to C_{(Y,\lambda)}$ is the moduli space, constructed by Mantovan, of isomorphisms

$$X_i \simeq Y_{i,C_{(Y,\lambda)}}$$

compatible with the polarisations up to a scalar in $\mathbb{Z}_p^\times$, and $\mathrm{Ig}_{\mathrm{CS},\lambda} \to C_{(Y,\lambda)}$ is the moduli space constructed by Caraiani–Scholze of isomorphisms

$$X \simeq Y_{C_{(Y,\lambda)}},$$

compatible with the polarisations up to a scalar in $\mathbb{Z}_p^\times$. Let $\mathbf{Aut}_\lambda(Y)$ be the group scheme of automorphisms of Y that preserve the polarisation λ up to a $\mathbb{Z}_p^\times$-scalar and let $\underline{\mathbf{Aut}_\lambda(Y)(\overline{\mathbb{F}}_p)}$ be its pro-étale group scheme of connected components.

It follows from work of Mantovan, [56], that $C_{(Y,\lambda)}$ is smooth and that $\mathrm{Ig}_{\mathrm{M},\lambda} \to C_{(Y,\lambda)}$ is a pro-étale torsor for $\underline{\mathbf{Aut}_\lambda(Y)(\overline{\mathbb{F}}_p)}$. Caraiani and Scholze, [9], prove that $\mathrm{Ig}_{\mathrm{CS},\lambda}$ is perfect and that the map $\mathrm{Ig}_{\mathrm{CS},\lambda} \to \mathrm{Ig}_{\mathrm{M},\lambda}$ identifies it with the perfection of $\mathrm{Ig}_{\mathrm{M},\lambda}$. This implies that $\mathrm{Ig}_{\mathrm{CS},\lambda} \to C_{(Y,\lambda)}$ is faithfully flat and thus a torsor for $\mathbf{Aut}_\lambda(Y)$. Moreover they prove that the action of $\mathbf{Aut}_\lambda(Y)$ on $\mathrm{Ig}_{\mathrm{CS},\lambda}$ extends to an action of $\mathbf{Aut}_\lambda(\tilde{Y})$, the group of automorphisms of $\tilde{Y}$ preserving the polarisation up to a $\mathbb{Q}_p^\times$-scalar.

We note that the following diagram is Cartesian

$$\begin{array}{ccc} \mathrm{Ig}_{\mathrm{CS},\lambda} & \longrightarrow & \mathrm{Ig}_{\mathrm{M},\lambda} \\ \downarrow & & \downarrow \\ C^{\mathrm{perf}}_{(Y,\lambda)} & \longrightarrow & C_{(Y,\lambda)}, \end{array} \tag{5.3.1}$$

where $C^{\mathrm{perf}}_{(Y,\lambda)}$ denotes the perfection of $C_{(Y,\lambda)}$.

5.4. **Igusa varieties for Shimura varieties of Hodge type.** Let $x \in \mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p)$ as above and choose an isomorphism

$$\mathbb{D}_{\mathrm{contr},x} \simeq V_{(p)} \otimes_{\mathbb{Z}_{(p)}} \breve{\mathbb{Z}}_p$$

taking $s_{\alpha,\mathrm{cris},x}$ to $s_\alpha \otimes 1$. This induces an isomorphism from $V^*_{(p)} \otimes \breve{\mathbb{Z}}_p$ to the covariant Dieudonné module $\mathbb{D}(Y)$ and thus gives us Frobenius invariant tensors $\{s_{\alpha,\mathrm{cris},x}\} \subset \mathbb{D}(Y)^{\otimes}$. Let $b \in G(\breve{\mathbb{Q}}_p) \subset \mathrm{GL}(V^*)(\breve{\mathbb{Q}}_p)$ be the element corresponding to the Frobenius on $\mathbb{D}(Y)[1/p]$. Under such an isomorphism, the Frobenius corresponds to an element $b = b_x \in G(\breve{\mathbb{Q}}_p) \subset \mathrm{GL}(V^*)(\breve{\mathbb{Q}}_p)$. In particular, we can apply the result of Section 4.3 and form the objects $\mathcal{H}_Y^{G,\circ} \subset \mathcal{H}_Y$ and $\mathbf{Aut}_G(\tilde{Y})$. We then consider the Dieudonné-Lie $\breve{\mathbb{Z}}_p$-algebra $\mathfrak{a}^+ = \mathbb{D}(\mathcal{H}_Y^{G,\circ})$ and $\mathfrak{a} = \mathfrak{a}^+ \otimes \breve{\mathbb{Q}}_p$. We assume that $p > n(\mathfrak{a})$, where $n(\mathfrak{a})$ is the nilpotency class of the Lie algebra $\mathfrak{a}$.

We have seen in Section 4.3 that $\mathfrak{a} \simeq \mathrm{Lie}\, U_{\nu_b}$, so that this condition only depends on the Newton stratum $\mathrm{Sh}_{G,U,[b]}$ that we are in. Indeed, σ-conjugate b give rise to conjugate Newton cocharacters ν_b. We have that $\mathbf{Aut}_G(\tilde{Y})^\circ \simeq \tilde{\Pi}(\mathfrak{a})$ and $\mathbf{Aut}_G(Y)^\circ \simeq \Pi(\mathfrak{a}^+)$. by the discussion in Section 4.3.

Hamacher and Kim define an Igusa variety $\mathrm{Ig}_{\mathrm{CS}} \to C^{\mathrm{perf}}_{G,[\![b]\!]}$ by pulling back $\mathrm{Ig}_{\mathrm{CS},\lambda} \to C^{\mathrm{perf}}_{(Y,\lambda)}$ and then taking the closed subset where the universal isomorphism

$$X \simeq Y_{C_{(Y,\lambda)}}$$

preserves the tensors on geometric points, see Section 5.1 and Lemma 5.1.1 of [34]. They then prove that $\mathrm{Ig}_{\mathrm{CS}} \to C^{\mathrm{perf}}$ is a pro-étale torsor for $\underline{\mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p)} \subseteq \underline{\mathbf{Aut}_\lambda(Y)(\overline{\mathbb{F}}_p)}$ and that $\mathrm{Ig}_{\mathrm{CS}}$ is stable under the action of $\mathbf{Aut}_G(\tilde{Y}) \subseteq \mathbf{Aut}(\tilde{Y})$. Using the invariance of the étale site under perfections, we get a pro-étale torsor $\mathrm{Ig}_{\mathrm{M}} \to C$ sitting in a Cartesian diagram

$$\begin{tikzcd} \mathrm{Ig}_{\mathrm{CS}} \arrow[r] \arrow[d] & \mathrm{Ig}_{\mathrm{M}} \arrow[d] \\ C^{\mathrm{perf}}_{G,[\![b]\!]} \arrow[r] & C_{G,[\![b]\!]}, \end{tikzcd}$$

that fits in a commutative cube with (5.3.1). The map $\mathrm{Ig}_{\mathrm{CS}} \to C_{G,[\![b]\!]}$ is faithfully flat since $\mathrm{Ig}_{\mathrm{CS}} \to C^{\mathrm{perf}}$ is a pro-étale cover and $C_{G,[\![b]\!]}$ is smooth. We want to prove that it is actually an fpqc torsor under $\mathbf{Aut}_G(Y)$. We first prove it at the level of infinitesimal neighbourhoods (defined as in Section 0AIX of [73]).

Lemma 5.4.1. *If $y \in \mathrm{Ig}_{\mathrm{CS}}(\overline{\mathbb{F}}_p)$ is an $\overline{\mathbb{F}}_p$-point over $x \in C_{G,[\![b]\!]}(\overline{\mathbb{F}}_p)$, then $\mathrm{Ig}_{\mathrm{CS}}^{/y} \to C_{G,[\![b]\!]}^{/x}$ is a torsor under the formal group $\mathbf{Aut}^\circ_G(Y)$.*

Proof. In Section 5.2 of [34], Hamacher and Kim show that the formal completion at a point of $\mathrm{Ig}_{\mathrm{CS}}$ is isomorphic to $\mathbf{Aut}^\circ_G(\tilde{Y})$ and that the natural action of $\mathbf{Aut}^\circ_G(\tilde{Y})$ on $\mathrm{Ig}_{\mathrm{CS}}$ corresponds to the multiplication map under this isomorphism. Moreover, the morphism $\mathrm{Ig}_{\mathrm{CS}}^{/y} \to C_{G,[\![b]\!]}^{/x}$ corresponds to the restriction of the action map (constructed in Theorem 4.3.1 of [47])

$$\mathbf{Aut}^\circ_G(\tilde{Y}) \times \mathrm{Sh}^{/x}_{G,U,[b]} \to \mathrm{Sh}^{/x}_{G,U,[b]}$$

to the closed point $x \in \mathrm{Sh}^{/x}_{G,U,[b]}$. Theorem 5.1.3 of [47] tells us that the scheme-theoretic image of this restriction

$$\mathbf{Aut}^\circ_G(\tilde{Y}) \to \mathrm{Sh}^{/x}_{G,U,[b]}$$

is $C^{/x}_{G,[\![b]\!]} \subset \mathrm{Sh}^{/x}_{G,U,[b]}$ and in fact this identifies $\mathbf{Aut}^{\circ}_{G}(\tilde{Y})$ with the perfection of $C^{/x}_{G,[\![b]\!]}$. If we apply this to $(G,X) = (\mathcal{G}_V, \mathcal{H}_V)$, the fact that

$$\mathrm{Ig}_{\mathrm{CS},\lambda} \to \mathrm{Ig}_{\mathrm{M},\lambda}$$

is a torsor for $\mathbf{Aut}^{\circ}_{\lambda}(Y)$ (plus the fact that $\mathrm{Ig}_{\mathrm{M},\lambda} \to C_{(Y,\lambda)}$ is pro-étale) tells us that

$$C^{/x}_{(Y,\lambda)} \simeq \frac{\mathbf{Aut}^{\circ}_{\lambda}(\tilde{Y})}{\mathbf{Aut}^{\circ}_{\lambda}(Y)}.$$

Since $\mathbf{Aut}^{\circ}_{G}(Y)$ is the intersection of $\mathbf{Aut}^{\circ}_{\lambda}(Y)$ with $\mathbf{Aut}_{G}(\tilde{Y})$, it follows similarly that $\mathbf{Aut}^{\circ}_{G}(Y) \subseteq \mathbf{Aut}^{\circ}_{G}(\tilde{Y})$ is the stabiliser of $x \in C^{/x}_{G,[\![b]\!]}$. Therefore, we get a monomorphism of formal schemes

$$\frac{\mathbf{Aut}^{\circ}_{G}(\tilde{Y})}{\mathbf{Aut}^{\circ}_{G}(Y)} \to C^{/x}_{G,[\![b]\!]},$$

both of which are formally smooth of dimension $\langle 2\rho, \nu_b \rangle$. Indeed for the left hand side this is Proposition 4.2.13, while for the right hand side it is Corollary 5.3.1 of [47]. Lemma 2.1.7 tells us that the map is a closed immersion and therefore we deduce that is an isomorphism. $\square$

Proposition 5.4.2. *The map* $\mathrm{Ig}_{\mathrm{CS}} \to C_{G,[\![b]\!]}$ *is an fpqc torsor under* $\mathbf{Aut}_{G}(Y)$*.*

Proof. Since $\mathrm{Ig}_{\mathrm{CS}} \to C_{G,[\![b]\!]}$ is faithfully flat, it suffices to prove that it is a quasi-torsor under $\mathbf{Aut}_{G}(Y)$. In other words, we want to show that the action map

$$\mathbf{Aut}_{G}(Y) \times \mathrm{Ig}_{\mathrm{CS}} \to \mathrm{Ig}_{\mathrm{CS}} \times_{C_{G,[\![b]\!]}} \mathrm{Ig}_{\mathrm{CS}} \tag{5.4.1}$$

is an isomorphism. This map is clearly a homeomorphism because $\mathrm{Ig}_{\mathrm{M}} \to C_{G,[\![b]\!]}$ is an $\underline{\mathbf{Aut}_{G}(Y)(\overline{\mathbb{F}}_p)}$-torsor and both $\mathrm{Ig}_{\mathrm{CS}} \to \mathrm{Ig}_{\mathrm{M}}$ and $\mathbf{Aut}_{G}(\tilde{Y}) \to \underline{\mathbf{Aut}_{G}(Y)(\overline{\mathbb{F}}_p)}$ are universal homeomorphisms.

It follows from Lemma 5.2.3 of [34] that when Z is either the source or the target of (5.4.1) and $z \in Z(\overline{\mathbb{F}}_p)$ is an $\overline{\mathbb{F}}_p$-point over x, then $Z^{/z}$ is pro-represented by the formal spectrum of the I-adic completion of $\mathcal{O}_{Z,x}$, where I is the maximal ideal of $\mathcal{O}_{C_{G,[\![b]\!]},x}$. Moreover, they prove that $\mathcal{O}_{Z,x} \to \mathcal{O}^{\wedge,I}_{Z,x}$ is faithfully flat.

It follows from Lemma 5.4.1 that the action map (5.4.1) is flat at all closed points and therefore it is flat by (7) of Lemma 00HT of [73]. We moreover know that the action map is a closed immersion because this is true in the Siegel case, so the action map is a surjective flat closed immersion and therefore an isomorphism (see Lemma 04PW of [73]). $\square$

5.5. **Deformation theory of central leaves.** If we let the notation be as above then Chai and Oort prove that the deformation theory of $C_{(Y,\lambda)}$ is completely determined by the $\mathbf{Aut}_{\lambda}(Y)$-torsor $\mathrm{Ig}_{\mathrm{CS},\lambda} \to C_{(Y,\lambda)}$. Let $\mathbf{Def}_{\mathrm{sus}}(Y,\lambda)$ denote the deformation space (considered as a functor on the category of Artin local rings over $\overline{\mathbb{F}}_p$ with residue field $\overline{\mathbb{F}}_p$) of the trivial $\mathbf{Aut}_{\lambda}(Y)$-torsor over $\overline{\mathbb{F}}_p$. Then Theorem 4.3 of [15] states that the natural map

$$C^{/x}_{(Y,\lambda)} \to \mathbf{Def}_{\mathrm{sus}}(Y,\lambda)$$

is an isomorphism of formal schemes. They prove this by showing that $\mathbf{Def}_{\mathrm{sus}}(Y,\lambda)$ is formally smooth of the same dimension as $C^{/x}_{(Y,\lambda)}$. We will need a version of their argument later, so we'll give an axiomatic treatment of it. Note that the deformation space of the trivial $\mathbf{Aut}_{\lambda}(Y)$-torsor over $\overline{\mathbb{F}}_p$ is (non-canonically) isomorphic to the deformation space of the trivial $\mathbf{Aut}^{\circ}_{\lambda}(Y)$-torsor over $\overline{\mathbb{F}}_p$ because torsors for pro-étale group schemes have trivial deformation theory.

Proposition 5.5.1. *Let $P/\overline{\mathbb{F}}_p$ be a group scheme that is an inverse limit of finite flat group schemes $\{P_i\}_{i\in\mathbb{Z}_{\geq 1}}$, equipped with a finite exhaustive and separated filtration by closed subgroups $\mathrm{Fil}^\bullet_P$. Assume that for each k there is a p-divisible group $\mathbb{X}_k$ such that*

$$\frac{\mathrm{Fil}^{k+1}_P}{\mathrm{Fil}^k_P} \simeq T_p\mathbb{X}_k,$$

Let $\mathrm{Fil}^\bullet_{P_i}$ be the induced filtration of P_i, and assume that there is an isomorphism of inverse system

$$\left\{\frac{\mathrm{Fil}^{k+1}_{P_i}}{\mathrm{Fil}^k_{P_i}}\right\}_{i\in\mathbb{Z}_{\geq 1}} \simeq \left\{\mathbb{X}_k[p^i]\right\}_{i\in\mathbb{Z}_{\geq 1}}.$$

Then the formal deformation space of the trivial P-torsor is formally smooth of dimension

$$\sum_k \dim \mathbb{X}_k.$$

Proof. Let $R' \to R$ be a square-zero extension of Artinian local rings with over $\overline{\mathbb{F}}_p$ with kernel I, and write $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$. Let T be a P-torsor over S together with a trivialisation $\alpha : T \times_{S_0} \operatorname{Spec} \overline{\mathbb{F}}_p \simeq P$. We need to show that there is a P-torsor T' over S' lifting T. Since T is the inverse limit of the P_i-torsors T_i induced from T via $P \to P_i$, it suffices to show that the T_i lift compatibly to S'.

For this, we will use the deformation theory for torsors under flat group schemes locally of finite presentation from Section 2.4 of Chapter VII of [43]. It follows from loc. cit. that there is a complex $\ell_{T_n/S}$ of $\mathcal{O}_S$-modules of amplitude contained in $[-1, 0]$, called the *co-Lie complex*. Theorem 2.4.4 of loc. cit. tells us that there is an obstruction element

$$o(T_i, R' \to R) \in H^2(\operatorname{Spec} \overline{\mathbb{F}}_p, \ell^\vee_{T_i/S} \otimes_{\overline{\mathbb{F}}_p} I),$$

which vanishes precisely when T_n lifts to S'. Moreover the set of all such lifts is a torsor under

$$H^1(\operatorname{Spec} \overline{\mathbb{F}}_p, \ell^\vee_{P_i/\overline{\mathbb{F}}_p} \otimes_{\overline{\mathbb{F}}_p} I).$$

Here $\ell^\vee_{P_i/\overline{\mathbb{F}}_p} =: \nu_{P_i}$ is the dual of $\ell_{P_i/\overline{\mathbb{F}}_p}$. The obstruction element vanishes for cohomological degree reasons, and so the torsor T_i lifts to S'. In order to show that our deformation space is formally smooth, we now need to show that

$$H^1(\operatorname{Spec} \overline{\mathbb{F}}_p, \ell^\vee_{P_{i+1}/\overline{\mathbb{F}}_p} \otimes_{\overline{\mathbb{F}}_p} I) \to H^1(\operatorname{Spec} \overline{\mathbb{F}}_p, \ell^\vee_{P_i/\overline{\mathbb{F}}_p} \otimes_{\overline{\mathbb{F}}_p} I)$$

is surjective for all i, so that we can pick a compatible sequence of lifts of the T_i to S'. In fact we will show that this map is an isomorphism for all i, so that we can compute the dimension in the case of $i = 1$. To show that this cohomology group does not depend on i, we will show that ν_{P_i} does not depend on i. Proposition 3.3.1 of Chapter VII of [43] tells us that a short exact sequence of flat locally of finite presentation group schemes leads to a distinguished triangle between their co-Lie complexes. Using the filtration $\mathrm{Fil}^k_{P_i}$, we see that it suffices to show that $\nu_{\mathbb{X}_k[p^i]}$ does not depend on i, but this is Proposition 2.2.1.c of [44].

To compute the dimension of our deformation functor, we note that Proposition 2.2.1.c of [44] also tells us that for a p-divisible group $\mathbb{X}$ over $\overline{\mathbb{F}}_p$, the dimension of

$$H^1(\operatorname{Spec} \overline{\mathbb{F}}_p, \ell^\vee_{\mathbb{X}[p]})$$

is equal to the dimension of the tangent space of X, and therefore equal to the dimension of X; the proposition follows. ∎

5.5.2. There is a canonical $\mathbf{Aut}_\lambda(Y)$-torsor over $C^{/x}_{(Y,\lambda)}$ given by the base-change of $\mathrm{Ig}_{\mathrm{CS},\lambda}$.

Lemma 5.5.3. *The induced map*

$$C^{/x}_{(Y,\lambda)} \to \mathbf{Def}_{\mathrm{sus}}(Y,\lambda).$$

is an isomorphism.

Proof. It suffices to show that it is a closed immersion because both sides are formally smooth of the same dimension. Indeed, the dimension of the right hand side can be computed using Proposition 5.5.1 and this agrees with the dimension of the left hand side, see [64]. There is a natural closed immersion

$$C^{/x}_{(Y,\lambda)} \to \mathbf{Def}(Y)$$

to the deformation space of (Y,λ), given by Serre–Tate theory. There is a subfunctor of the deformation space of (Y,λ), given by those deformations that are fpqc locally isomorphic to (Y,λ). This subfunctor can be identified with $\mathbf{Def}_{\mathrm{sus}}(Y,\lambda)$ because the universal $\mathbf{Aut}_\lambda(Y)$-torsor determines the universal p-divisible group. Moreover the subfunctor is automatically closed by Lemma 2.1.7. □

In the Hodge type case we define $\mathbf{Def}_{\mathrm{sus}}(Y,G)$ to be the deformation space of the trivial $\mathbf{Aut}_G(Y)$-torsor, then there is a commutative diagram

$$\begin{tikzcd} C^{/x}_{G,[\![b]\!]} \arrow[r] \arrow[d, hook] & \mathbf{Def}_{\mathrm{sus}}(Y,G) \arrow[d, "i"] \\ C^{/x}_{(Y,\lambda)} \arrow[r, hook] & \mathbf{Def}_{\mathrm{sus}}(Y,\lambda). \end{tikzcd} \tag{5.5.1}$$

Lemma 5.5.4. *The map $C^{/x}_{G,[\![b]\!]} \to \mathbf{Def}_{\mathrm{sus}}(Y,G)$ is an isomorphism.*

Proof. We start by proving that i is a closed immersion. For this, we note that by Lemma 2.1.7 it suffices to prove that it is a monomorphism. Equivalently, we need to show that given an Artin local ring R together with a $\mathbf{Aut}_\lambda(Y)$-torsor $\mathcal{P}$ over R, the set of reductions of structure group of P to $\mathbf{Aut}_G(Y)$, compatible with the identification

$$P \otimes_R \overline{\mathbb{F}}_p = \mathbf{Aut}_\lambda(Y),$$

is either empty or consists of a single element. This set is either empty or it has a transitive action of the group $\mathbf{Aut}_\lambda(Y)^\circ(R)$. But this group is trivial since

$$\mathbf{Aut}^\circ_\lambda(Y) \simeq \operatorname{Spec} \overline{\mathbb{F}}_p[X_1^{1/p^\infty}, \cdots, X_n^{1/p^\infty}]/(X_1, \cdots, X_n)$$

has no non-trivial sections over Noetherian test objects.

Proposition 5.5.1 together with Lemma 4.3.1 tells us that $\mathbf{Def}_{\mathrm{sus}}(Y,G)$ is formally smooth of dimension $\langle 2\rho, \nu_b\rangle$, which is equal to the dimension of $C_{G,[\![b]\!]}$ by Corollary 5.3.1 of [47]. It follows from the diagram (5.5.1) that the map $C^{/x}_{G,[\![b]\!]} \to \mathbf{Def}_{\mathrm{sus}}(Y,G)$ is a closed immersion between two formally smooth formal schemes of the same dimension, and therefore it is an isomorphism. □

Remark 5.5.5. If $Y/\overline{\mathbb{F}}_p$ is a completely slope divisible p-divisible group, then one can prove that the deformation space $\mathbf{Def}_{\mathrm{sus}}(Y)$ of the trivial $\mathbf{Aut}(Y)^\circ$-torsor admits a closed immersion to the deformation space $\mathbf{Def}(Y)$ of Y. Its image is identified with the subspace of deformations of Y that are fpqc locally isomorphic to the constant deformation of Y. It follows from Section 5 of [47] that there is an action of $\mathbf{Aut}(\tilde{Y})^\circ$ on $\mathbf{Def}_{\mathrm{sus}}(Y)$ which gives an identification

$$\mathbf{Def}_{\mathrm{sus}}(Y) \simeq \frac{\mathbf{Aut}(\tilde{Y})^\circ}{\mathbf{Aut}(Y)^\circ}.$$

Remark 5.5.6. If $Y = Y_1 \oplus Y_2$ then $\mathbf{Aut}(\tilde{Y})^\circ$ is isomorphic to $\widetilde{\mathcal{H}}_{Y_1,Y_2}$ and $\mathbf{Aut}(Y)^\circ \simeq T_p\mathcal{H}_{Y_1,Y_2}$ so that $\mathbf{Def}_{\mathrm{sus}}(Y) \simeq \mathcal{H}_{Y_1,Y_2}$. This gives $\mathbf{Def}_{\mathrm{sus}}(Y)$ the structure of a p-divisible formal group.

5.6. **Strongly Tate-linear subspaces.** Let $Y/\overline{\mathbb{F}}_p$ be a completely slope divisible p-divisible group and let X be the universal p-divisible group over the sustained deformation space $\mathbf{Def}_{\mathrm{sus}}(Y)$. Let $Z \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$ be a formally smooth closed subscheme, then the monodromy group $G(\mathcal{M}_Z)$ of the isocrystal $\mathcal{M} = \mathbb{D}(X)[\frac{1}{p}]$ restricted to Z, with respect to the closed point of Z, has a natural inclusion

$$\operatorname{Lie} G(\mathcal{M}_Z) \subseteq \mathbb{D}(\mathcal{H}_Y^\circ)[\tfrac{1}{p}] \subseteq \mathbb{D}(\mathcal{H}_Y)[\tfrac{1}{p}] = \operatorname{Lie} \mathrm{GL}(\mathbb{D}(Y))[\tfrac{1}{p}].$$

because X admits a slope filtration with constant graded parts. Since $\mathcal{M}$ has the structure of an F-isocrystal we get, by Section 2.2 of [21], an isomorphism

$$F^*G(\mathcal{M}_Z) \to G(\mathcal{M}_Z),$$

which induces an isomorphism

$$F^* \operatorname{Lie} G(\mathcal{M}_Z) \to \operatorname{Lie} G(\mathcal{M}_Z)$$

compatible with the F-structure on $\mathbb{D}(\mathcal{H}_Y^\circ)[\frac{1}{p}]$. In particular

$$\mathfrak{b} := \operatorname{Lie} G(\mathcal{M}_Z) \subseteq \mathbb{D}(\mathcal{H}_Y^\circ)[\tfrac{1}{p}] =: \mathfrak{a}_Y$$

is a sub-F-isocrystal stable under the Lie bracket.

Note that $\mathfrak{a}_Y^+ := \mathbb{D}(\mathcal{H}_Y^\circ)$ is a nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra by Example 4.2.2, thus we get a nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra

$$\mathfrak{b}^+ = \mathfrak{a}_Y^+ \cap \mathfrak{b}.$$

5.6.1. Let $\mathfrak{b} \subseteq \mathbb{D}(\mathcal{H}_Y^\circ)[\frac{1}{p}]$ be an F-stable Lie subalgebra as above. There is a closed immersion of Lie $\underline{\mathbb{Q}_p}$-algebras

$$\tilde{\mathbb{X}}(\mathfrak{b}) \hookrightarrow \tilde{\mathbb{X}}(\mathfrak{a}_Y) = \widetilde{\mathcal{H}}_Y^\circ.$$

Using the BCH-formula we get a closed group homomorphism

$$\tilde{\Pi}(\mathfrak{b}) \hookrightarrow \tilde{\Pi}(\mathfrak{a}_Y) = \mathbf{Aut}(\tilde{Y})^\circ$$

and similarly if $p > n$ we have

$$\Pi(\mathfrak{b}^+) \subseteq \Pi(\mathfrak{a}_Y^+) = \mathbf{Aut}(Y)^\circ.$$

This means that we get a formally smooth closed formal subscheme

$$Z(\mathfrak{b}^+) = \frac{\tilde{\Pi}(\mathfrak{b})}{\Pi(\mathfrak{b}^+)} \hookrightarrow \mathbf{Def}_{\mathrm{sus}}(Y)$$

and the same proof as in the previous section shows that $Z(\mathfrak{b}^+)$ can be identified with the deformation space of the trivial $\Pi(\mathfrak{b}^+)$-torsor. To be precise, Lemma 5.5.4 tells us that the natural map to the deformation space of the trivial $\Pi(\mathfrak{b}^+)$-torsor is a closed immersion, and then it is an isomorphism because source and target are formally smooth of the same dimension.

Therefore the subspace $Z(\mathfrak{b}^+)$ is a strongly Tate-linear subspace of $\mathbf{Def}_{\mathrm{sus}}(Y)$ in the sense of Chai–Oort, see Definition 6.2 of [15]. This requires a bit of translation since they work with projective systems of torsors for a projective system of finite flat group schemes and we are instead working with torsors for the inverse limit of the projective system of finite flat group schemes.

Conjecture 5.6.2. *Let $Z \hookrightarrow \mathbf{Def}_{\mathrm{sus}}(Y)$ be a closed immersion. For each F-stable Lie subalgebra $\mathfrak{b} \subset \mathfrak{a}_Y$ there is an inclusion $Z \subseteq Z(\mathfrak{b}^+)$ if and only if* $\operatorname{Lie} G(\mathcal{M}_Z) \subseteq \mathfrak{b}$. *In particular* $\operatorname{Lie} G(\mathcal{M}_{Z(\mathfrak{b}^+)}) = \mathfrak{b}$.

Remark 5.6.3. If $\mathfrak{b} = \mathbb{D}(\mathcal{H}_Y^{G,\circ})$ for a Shimura variety of Hodge type (G, X) with Y lying in a $\mathbb{Q}$-non-basic Newton stratum (see Definition 8.3.1) then the last assertion in the conjecture is true. Indeed the unipotent radical of the monodromy group

$$\operatorname{Mon}(C_{G,[\![b]\!]}, \mathcal{M})$$

is isomorphic to U_{ν_b} by Corollary 3.3.5 of [39][4], which uses [22, 23], and then Theorem 3.4.4 allows us to conclude.

As a special case of this if Y has height h and dimension d, then $\mathbf{Def}_{\mathrm{sus}}(Y)$ can be realised as the complete local ring of a central leaf in a PEL type unitary Shimura variety of signature $(h-d, d)$ associated to an imaginary quadratic field E in which p splits. In particular we know that the monodromy group of $\mathcal{M}/\mathbf{Def}_{\mathrm{sus}}(Y)$ is isomorphic to the unipotent group corresponding to the nilpotent Lie algebra $\mathbb{D}(\mathcal{H}_Y^\circ)[\frac{1}{p}] = \mathfrak{a}_Y$.

5.6.4. There is a $\breve{\mathbb{Z}}_p$-algebra structure on $\mathbb{D}(\mathcal{H}_Y)$ and $1+\mathbb{D}(\mathcal{H}_Y^\circ)$ is a subgroup , defining a unipotent algebraic group over $\breve{\mathbb{Z}}_p$. We can also consider $\mathbb{D}(\mathcal{H}_Y^\circ) \subseteq \mathbb{D}(\mathcal{H}_Y)$ as a Lie subalgebra, and consider the associated unipotent algebraic group. If Y is completely slope divisible and $p > n\left(\mathbb{D}(\mathcal{H}_Y^\circ)[\frac{1}{p}]\right)$, or equivalently if p is greater than the number of slopes of Y minus one, then these two constructions are the same.

Similarly given $\mathfrak{b} \subseteq \mathfrak{a}_Y$ we get a nilpotent $\breve{\mathbb{Z}}_p$-Lie algebra $\mathfrak{b}^+ = \mathfrak{a}_Y \cap \mathfrak{b}$ whose exponential defines a unipotent algebraic group $U(\mathfrak{b}^+)$ over $\breve{\mathbb{Z}}_p$. Its generic fibre is a unipotent group $U(\mathfrak{b})$ which only depends on $\mathfrak{b}$ and comes with a closed immersion

$$U(\mathfrak{b}) \subseteq 1 + \mathbb{D}(\mathcal{H}_Y^\circ).$$

6. Local monodromy of strongly Tate-linear subvarieties

6.1. **Introduction.** In this section we will prove half of Conjecture 5.6.2. Let $Y/\overline{\mathbb{F}}_p$ be a completely slope divisible p-divisible group and let X be the universal p-divisible group over the sustained deformation space $\mathbf{Def}_{\mathrm{sus}}(Y)$. Let $\mathfrak{a}^+ = \mathbb{D}(\mathcal{H}_Y^\circ)$ be the Dieudonné–Lie algebra associated to the internal-hom p-divisible group of Y and let $\mathfrak{b} \subset \mathfrak{a}$ be an F-stable Lie subalgebra with associated strongly Tate-linear subspace $Z := Z(\mathfrak{b}^+) \subset \mathbf{Def}_{\mathrm{sus}}(Y)$. Write $\mathcal{M} = \mathbb{D}(X)[\frac{1}{p}]$ for the isocrystal over $\mathbf{Def}_{\mathrm{sus}}(Y)$ coming from the Dieudonné-module of X.

Theorem 6.1.1. *The monodromy group $G(\mathcal{M}_Z)$ of the restriction of $\mathcal{M}$ to Z is contained in $U(\mathfrak{b})$.*

[4]Hypothesis 2.3.1 of [39] which is assumed to hold in the statement of Corollary 3.3.5 of [39] is true because K_p is hyperspecial, see Lemma 2.3.2 of [39].

In the proof we will make use of the Cartier–Witt stacks of Bhatt–Lurie [3] associated to quasisyntomic schemes of characteristic p. Given such a scheme X, there is a p-adic formal stack $X^{\mathbb{\Delta}}$, the *prismatisation* of X, such that coherent crystals on X are the same as coherent sheaves on $X^{\mathbb{\Delta}}$. In Section 6.2 we will give a more detailed overview of this construction and its properties.

We will now give a sketch of the proof. Consider the $\mathbf{Aut}(Y)$-torsor

$$\mathrm{Isom}(X, Y_{\mathbf{Def}_{\mathrm{sus}}(Y)}) \to \mathbf{Def}_{\mathrm{sus}}(Y) \tag{6.1.1}$$

over $\mathbf{Def}_{\mathrm{sus}}(Y)$. The locally free crystal $\mathcal{M}^+ = \mathbb{D}(X)$ defines a vector bundle $\mathcal{V}^+$ over the *prismatisation* $\mathbf{Def}_{\mathrm{sus}}(Y)^{\mathbb{\Delta}}$[5]. This vector bundle has an associated frame bundle which we will write suggestively as

$$\mathrm{Isom}(\mathcal{V}^+, \mathbb{D}(Y)_{\mathbf{Def}_{\mathrm{sus}}(Y)^{\mathbb{\Delta}}}). \tag{6.1.2}$$

If we apply the prismatisation functor to the map (6.1.1), then we get a morphism

$$\mathrm{Isom}(X, Y_{\mathbf{Def}_{\mathrm{sus}}(Y)})^{\mathbb{\Delta}} \to \mathbf{Def}_{\mathrm{sus}}(Y)^{\mathbb{\Delta}},$$

which will be a torsor for the p-adic formal group $\mathbf{Aut}(Y)^{\mathbb{\Delta}}$ (this will follow from Lemma 6.3.2). Dieudonné-theory gives us a homomorphism of group schemes over $\mathrm{Spf}\, \mathbb{Z}_p$

$$\mathbf{Aut}(Y)^{\mathbb{\Delta}} \to \mathrm{Aut}(\mathbb{D}(Y)) \tag{6.1.3}$$

and a morphism

$$\mathrm{Isom}(X, Y_{\mathbf{Def}_{\mathrm{sus}}(Y)})^{\mathbb{\Delta}} \to \mathrm{Isom}(\mathcal{V}^+, \mathbb{D}(Y)_{\mathbf{Def}_{\mathrm{sus}}(Y)^{\mathbb{\Delta}}}),$$

which is $\mathbf{Aut}(Y)^{\mathbb{\Delta}}$-equivariant via the homomorphism (6.1.3). The right hand side roughly speaking parametrises all isomorphisms between $\mathcal{V}^+$ and $\mathbb{D}(Y)_{\mathbf{Def}_{\mathrm{sus}}(Y)^{\mathbb{\Delta}}}$, while the left hand side parametrises those isomorphisms that are compatible with the F-structures.

So how does this help us? Well over $Z = Z(\mathfrak{b}^+)$ the torsor (6.1.1) has a reduction to a $\Pi(\mathfrak{b}^+)$-torsor by construction. Feeding this fact into the prismatisation machinery will give us a reduction of the torsor (6.1.2) to a $U(\mathfrak{b}^+)$-torsor. If we apply the Tannakian perspective on torsors and invert p, then this will exactly give us a closed immersion

$$G(\mathcal{M}_Z) \to U(\mathfrak{b}),$$

which is exactly what we want to prove.

6.2. **Cartier–Witt stacks.** Let us briefly recall the main properties of the *Cartier–Witt stacks* from [3]. We will deal only with *quasisyntomic schemes* X over $\mathbb{F}_p$, as in Definition 2.2.3.

6.2.1. Write $\mathrm{Nilp}_{\mathbb{Z}_p}^{\mathrm{op}} \subseteq \mathrm{Alg}_{\mathbb{Z}_p}^{\mathrm{op}}$ for the full subcategory of p-nilpotent algebras with the fpqc topology. A *p-adic formal stack* is a groupoid valued functor $\mathcal{F}$ on $\mathrm{Nilp}_{\mathbb{Z}_p}^{\mathrm{op}}$ whose diagonal is representable by formal algebraic spaces and which admits an fpqc cover $\mathcal{X} \to \mathcal{F}$, where $\mathcal{X}$ is a p-adic formal algebraic space over $\mathrm{Spf}\, \mathbb{Z}_p$ (cf. [73, Definition 0AIM]).

Bhatt and Lurie define a prismatisation functor

$$X \mapsto X^{\mathbb{\Delta}}$$

which goes from the category of quasisyntomic $\mathbb{F}_p$-schemes to the category of p-adic formal stacks endowed with an endomorphism $F : X^{\mathbb{\Delta}} \to X^{\mathbb{\Delta}}$, lifting the Frobenius on the special fibre[6].

[5]We pretend for now that our formal schemes are actually schemes, in the actual proof there is an additional algebraisation step.

[6]They also define a derived version of this functor, which we will not use in this text.

6.2.2. For every quasisyntomic scheme X, Proposition 8.15 of [3] tells us that there is an equivalence between the category of crystals in quasi-coherent $\mathcal{O}$-modules on the absolute prismatic site of X (or the absolute crystalline site by Example 4.7 of [5]) and quasi-coherent $\mathcal{O}$-modules on the Zariski site of X^{Δ}.

There are a few important properties of this functor that we will use.

– If $X = \operatorname{Spec} R$ is a semiperfect quasisyntomic scheme, then X^{Δ} is simply $\operatorname{Spf} A_{\mathrm{cris}}(R)$, the formal spectrum of Fontaine's ring of crystalline periods (this is Lemma 6.1 of [3], see Proposition 4.1.3 of [68] for a definition of $A_{\mathrm{cris}}(R)$). For example $(\operatorname{Spec} \mathbb{F}_p)^{\Delta} = \operatorname{Spf} \mathbb{Z}_p$.

– If $f : X \to Y$ is a quasisyntomic cover in the sense of Section 4 of [4], see Definition 2.2.4, then $f : X^{\Delta} \to Y^{\Delta}$ is an fpqc cover (this is Proposition 7.5 of [3]). For example, this means that $X^{\Delta} \to (\operatorname{Spec} \mathbb{F}_p)^{\Delta} = \operatorname{Spf} \mathbb{Z}_p$ is automatically flat if X is qrsp.

– The functor commutes with products and with fibre products in the case that the structure maps are flat and quasisyntomic, by Remark 8.9 of [3] and Proposition 7.5 of [3].

6.3. **Algebraisation.** Let $Z = Z(\mathfrak{b}) \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$ be a strongly Tate-linear formal closed subscheme of $\mathbf{Def}_{\mathrm{sus}}(Y)$ corresponding to a Dieudonné–Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{a}_Y$. The formal scheme Z is equal to $\operatorname{Spf} A$ for a complete Noetherian local ring A. Proposition 2.4.8 of [24] tells us that the category of Dieudonné isocrystals over $\operatorname{Spf} A$ is equivalent to the category of Dieudonné isocrystals over $\operatorname{Spec} A$. Moreover, we have that the $\Pi(\mathfrak{b}^+)$-torsor P_Z over $Z = \operatorname{Spf} A$ comes from a $\Pi(\mathfrak{b}^+)$-torsor P_Z^{alg} over $\operatorname{Spec} A$. Indeed P is an inverse limit of torsors under finite flat group schemes over Z, and those all algebraise because finite modules do functorially.

Notation 6.3.1. In Section 6 we will treat Z as the affine scheme $\operatorname{Spec} A$ rather than the formal scheme $\operatorname{Spf} A$ and we will simply write $P_Z \to Z$ for the algebraic torsor P_Z^{alg} defined above. The same applies to $\mathbf{Def}_{\mathrm{sus}}(Y)$.

Let $\mathcal{M}^+$ be the F-crystal over $\mathbf{Def}_{\mathrm{sus}}(Y)$ attached to the universal p-divisible group. We write $\mathcal{M}$ for the induced F-isocrystal and $\mathcal{M}_Z$ the restriction of $\mathcal{M}$ to Z. Write $i : z \hookrightarrow Z$ for the closed embedding of the closed point of Z.

Proof of Theorem 6.1.1. Let $P \to Z$ be the restriction of the universal $\mathbf{Aut}(Y)$-torsor and let $P_Z \to Z$ be the reduction of P to a $\Pi(\mathfrak{b}^+)$-torsor. The basic idea of the proof is to use descent of isocrystals along $P_Z \to Z$ to describe $\mathcal{M}_{P_Z}$ as a constant isocrystal equipped with a descent datum (or equivalently an $\Pi(\mathfrak{b}^+)$-equivariant structure). However it seems quite hard to compare the group scheme $\Pi(\mathfrak{b}^+)$ over $\overline{\mathbb{F}}_p$ with the monodromy group of $\mathcal{M}$, which is an algebraic group over $\breve{\mathbb{Q}}_p$. This is where the Cartier–Witt stacks of [3] come in.

The Dieudonné module $\mathbb{D}(Y)$ of Y is a trivial vector bundle on $\overline{\mathbb{F}}_p^{\Delta} = \operatorname{Spf}(\breve{\mathbb{Z}}_p)$ endowed with a Frobenius. We denote by $\mathrm{GL}(\mathbb{D}(Y))$ the p-adic formal group over $\operatorname{Spf}(\breve{\mathbb{Z}}_p)$ of $\breve{\mathbb{Z}}_p$-linear automorphisms of $\mathbb{D}(Y)$ (thus forgetting the F-structure). Let $U(\mathfrak{b}^+) \subset \mathrm{GL}(\mathbb{D}(Y))$ denote the inclusion of the p-adic completion of the unipotent group $U(\mathfrak{b}^+)$.

By the formalism of Cartier–Witt stacks, the crystal $\mathcal{M}^+$ corresponds to a vector bundle $\mathcal{V}^+$ on Z^{Δ}. In turn, this defines a formal $\mathrm{GL}(\mathbb{D}(Y))$-torsor

$$\operatorname{Isom}(\mathcal{V}^+, \mathbb{D}(Y)_{Z^{\Delta}}) \to Z^{\Delta}$$

over $Z^{\mathbb{\Delta}}$ which we denote by $\mathfrak{Q} \to Z^{\mathbb{\Delta}}$. On the other hand, $P_Z \to Z$ induces a $\Pi(\mathfrak{b}^+)^{\mathbb{\Delta}}$-torsor $P_Z^{\mathbb{\Delta}} \to Z^{\mathbb{\Delta}}$ of p-adic formal stacks by the following lemma.

Lemma 6.3.2. *Let T be a torsor over a quasisyntomic scheme S over $\mathbb{F}_p$ under a qrsp group scheme G. The prismatisation $G^{\mathbb{\Delta}}$ of G is a formal group scheme and $T^{\mathbb{\Delta}} \to S^{\mathbb{\Delta}}$ is a $G^{\mathbb{\Delta}}$-torsor of formal stacks.*

Proof. Since G is qrsp, the prismatisation is a p-adic formal scheme. In addition, since prismatisation of $\mathbb{F}_p$-schemes commutes with products, it follows that $G^{\mathbb{\Delta}}$ is a formal group scheme. The fact that prismatisation sends quasisyntomic covers to fpqc covers and commutes with fibre products when the structure maps are quasisyntomic covers tells us that $T^{\mathbb{\Delta}} \to S^{\mathbb{\Delta}}$ is a torsor for $G^{\mathbb{\Delta}}$. $\square$

6.3.3. We may apply Lemma 6.3.2 in our situation since the group scheme $\Pi(\mathfrak{b}^+)$ over $\overline{\mathbb{F}}_p$ is qrsp by Lemma 4.2.12. Write $\Pi(\mathfrak{b}^+) = \operatorname{Spec} R$ and consider the tautological element $g_{\text{univ}} \in \Pi(\mathfrak{b}^+)(R)$. This element corresponds to an automorphism

$$g_{\text{univ}} : Y_R \to Y_R,$$

and this induces an automorphism of Dieudonné-modules

$$\mathbb{D}(g_{\text{univ}}) : \mathbb{D}(Y) \otimes_{\breve{\mathbb{Z}}_p} A_{\text{cris}}(R) \to \mathbb{D}(Y) \otimes_{\breve{\mathbb{Z}}_p} A_{\text{cris}}(R).$$

This corresponds precisely to a $\operatorname{Spf} A_{\text{cris}}(R) = \Pi(\mathfrak{b}^+)^{\mathbb{\Delta}}$-point of $\mathrm{GL}(\mathbb{D}(Y))$, in other words, it corresponds to a map

$$\rho : \Pi(\mathfrak{b}^+)^{\mathbb{\Delta}} \to \mathrm{GL}(\mathbb{D}(Y)).$$

Lemma 6.3.4. *The image of ρ lands in the closed subgroup $U(\mathfrak{b}^+) \subseteq \mathrm{GL}(\mathbb{D}(Y))$. Moreover, the morphism ρ is a group homomorphism.*

Proof. The definition of $\Pi(\mathfrak{b}^+)$ tells us that g_{univ} has logarithm in $\tilde{\mathbb{X}}(\mathfrak{b})(R)$. Since the logarithm map commutes with the Dieudonné-module functor (because the functor commutes with composition and is additive) we see that $\mathbb{D}(g_{\text{univ}})$ has logarithm contained in

$$1 + \mathfrak{b} \otimes_{\breve{\mathbb{Q}}_p} \mathrm{B}^+_{\text{cris}}(R) \subset \operatorname{End}(\mathbb{D}(Y) \otimes_{\breve{\mathbb{Z}}_p} B^+_{\text{cris}}(R)).$$

Our assumption that $p > n(\mathfrak{a}) \geq n(\mathfrak{b})$ implies that the logarithm of $\mathbb{D}(g_{\text{univ}})$ in fact lies in

$$1 + \mathfrak{b}^+ \otimes_{\breve{\mathbb{Z}}_p} \mathrm{A}_{\text{cris}}(R) \subset \operatorname{End}(\mathbb{D}(Y) \otimes_{\breve{\mathbb{Z}}_p} A_{\text{cris}}(R)),$$

and therefore ρ factors through the unipotent group associated to $\mathfrak{b}^+$. The second claim of the lemma is that the following diagram commutes (where the vertical maps are the multiplication maps)

$$\begin{tikzcd}
\Pi(\mathfrak{b}^+)^{\mathbb{\Delta}} \times \Pi(\mathfrak{b}^+)^{\mathbb{\Delta}} \arrow[r, "\rho"] \arrow[d] & \mathrm{GL}(\mathbb{D}(Y)) \times \mathrm{GL}(\mathbb{D}(Y)) \arrow[d] \\
\Pi(\mathfrak{b}^+)^{\mathbb{\Delta}} \arrow[r] & \mathrm{GL}(\mathbb{D}(Y)).
\end{tikzcd}$$

For $i = 1, 2$ let $p_{i,\mathrm{GL}} : \mathrm{GL}(\mathbb{D}(Y)) \times \mathrm{GL}(\mathbb{D}(Y)) \to \mathrm{GL}(\mathbb{D}(Y))$ and $p_{i,\Pi} : \Pi(\mathfrak{b}^+)^{\mathbb{\Delta}} \times \Pi(\mathfrak{b}^+)^{\mathbb{\Delta}} \to \Pi(\mathfrak{b}^+)^{\mathbb{\Delta}}$ be the projection maps. Using the Yoneda lemma, it suffices to show the equality

$$p_{1,\mathrm{GL}}^* \mathbb{D}(g_{\text{univ}}) \circ p_{2,\mathrm{GL}}^* \mathbb{D}(g_{\text{univ}}) = \mathbb{D}(p_{1,\Pi}^* g_{\text{univ}} \circ p_{2,\Pi}^* g_{\text{univ}})$$

as elements of

$$\mathrm{GL}(\mathbb{D}(Y))\left(\Pi(\mathfrak{b}^+)^{\mathbb{\Delta}} \times \Pi(\mathfrak{b}^+)^{\mathbb{\Delta}}\right) = \mathbf{Aut}(\mathbb{D}(Y) \otimes_{\breve{\mathbb{Z}}_p} \mathrm{A}_{\mathrm{cris}}(R \otimes_{\overline{\mathbb{F}}_p} R).$$

But functoriality of Dieudonné-theory tells us that

$$\begin{aligned}\mathbb{D}(p_{1,\Pi}^* g_{\mathrm{univ}} \circ p_{2,\Pi}^* g_{\mathrm{univ}}) &= \mathbb{D}(p_{1,\Pi}^* g_{\mathrm{univ}}) \circ \mathbb{D}(p_{2,\Pi}^* g_{\mathrm{univ}}) \\ &= p_{1,\mathrm{GL}}^* \mathbb{D}(g_{\mathrm{univ}}) \circ p_{2,\mathrm{GL}}^* \mathbb{D}(g_{\mathrm{univ}}).\end{aligned}$$

□

6.3.5. There is an isomorphism

$$h_{\mathrm{univ}} : X_{P_Z} \to Y_{P_Z}$$

corresponding to the identity map $P_Z \to P_Z$ and applying the Dieudonné-theory functor we get an isomorphism

$$\mathcal{V}^+_{P_Z^{\mathbb{\Delta}}} \to \mathbb{D}(Y)_{P_Z^{\mathbb{\Delta}}},$$

which corresponds to a morphism

$$\sigma : P_Z^{\mathbb{\Delta}} \to \mathfrak{Q}.$$

Lemma 6.3.6. *The map σ is $\Pi(\mathfrak{b}^+)^{\mathbb{\Delta}}$-equivariant, where $\Pi(\mathfrak{b}^+)^{\mathbb{\Delta}}$ acts on $\mathfrak{Q}$ via ρ.*

Proof. We are trying to show that the following diagram commutes

$$\begin{tikzcd} \Pi(\mathfrak{b}^+)^{\mathbb{\Delta}} \times P_Z^{\mathbb{\Delta}} \arrow[r, "(\rho,\sigma)"] \arrow[d] & \mathrm{GL}(\mathbb{D}(Y)) \times \mathfrak{Q} \arrow[d] \\ P_Z^{\mathbb{\Delta}} \arrow[r, "\sigma"] & \mathfrak{Q}, \end{tikzcd}$$

where the vertical maps are given by the respective action maps. It suffices to prove that the diagram commutes on $\Pi(\mathfrak{b}^+)^{\mathbb{\Delta}} \times P_Z^{\mathbb{\Delta}}$-points, which we will write as $\mathrm{Spf}\, A_{\mathrm{cris}}(R) \times \mathrm{Spf}\, A_{\mathrm{cris}}(S)$. The identity map $R \otimes S \to R \otimes S$ corresponds to $(g_{\mathrm{univ}}^{\mathbb{\Delta}}, h_{\mathrm{univ}}^{\mathbb{\Delta}})$ and (ρ, σ) corresponds to $(\mathbb{D}(g_{\mathrm{univ}}), \mathbb{D}(h_{\mathrm{univ}}))$. The map to $P_Z^{\mathbb{\Delta}}$ corresponds to the composition $g_{\mathrm{univ}} \circ h_{\mathrm{univ}}$. The commutativity of the diagram is equivalent to the equality

$$\mathbb{D}(g_{\mathrm{univ}} \circ h_{\mathrm{univ}}) = \mathbb{D}(g_{\mathrm{univ}}) \circ \mathbb{D}(h_{\mathrm{univ}}),$$

which follows from functoriality of Dieudonné-theory. □

Since ρ factors through $U(\mathfrak{b}^+)$, we get a reduction of the $\mathrm{GL}\,(\mathbb{D}(Y))$-torsor $\mathfrak{Q} \to Z^{\mathbb{\Delta}}$ to a $U(\mathfrak{b}^+)$-torsor $\mathfrak{R} \to Z^{\mathbb{\Delta}}$ sitting between $P_Z^{\mathbb{\Delta}}$ and $\mathfrak{Q}$. We can associate to this the symmetric tensor functor

$$\Psi : \mathrm{Rep}_{\breve{\mathbb{Z}}_p}(U(\mathfrak{b}^+)) \to \mathrm{Vect}(Z^{\mathbb{\Delta}})$$

which sends $V \in \mathrm{Rep}_{\breve{\mathbb{Z}}_p}(U(\mathfrak{b}^+))$ to

$$\mathfrak{R} \times^{U(\mathfrak{b}^+)} (V \otimes_{\breve{\mathbb{Z}}_p} Z^{\mathbb{\Delta}}).$$

The tautological representation $U(\mathfrak{b}^+) \hookrightarrow \mathrm{GL}\,(\mathbb{D}(Y))$ is sent by Ψ to the vector bundle $\mathcal{V}^+$.

In order to pass to the generic fibre, we need the following result, which is a special case of the proposition stated in Section 6.4 of [74].

Lemma 6.3.7. *Let $\mathfrak{G}$ be a smooth group scheme over $\breve{\mathbb{Z}}_p$ with generic fibre $\mathfrak{G}_\eta$. Then every representation $\rho : \mathfrak{G}_\eta \to \mathrm{GL}(V)$, where V is a finite dimensional $\breve{\mathbb{Q}}_p$-vector space, extends to a representation $\mathfrak{G} \to \mathrm{GL}(\Lambda)$ for some $\breve{\mathbb{Z}}_p$-lattice $\Lambda \subseteq V$.*

Applying the lemma and passing to isogeny categories, we get an exact tensor functor

$$\Psi_{\breve{\mathbb{Q}}_p} : \mathrm{Rep}_{\breve{\mathbb{Q}}_p}(U(\mathfrak{b})) \to \mathrm{Vect}(Z^{\mathbb{\Delta}})[\tfrac{1}{p}]$$

sending the defining representation of $U(\mathfrak{b})$ to $\mathcal{V}$. We can compose this with the natural inclusion

$$\mathrm{Vect}(Z^{\mathbb{\Delta}})[\tfrac{1}{p}] \hookrightarrow \mathrm{Isoc}(Z)$$

and apply Tannaka duality to get a morphism of group schemes

$$G(\mathcal{M}_Z) \to U(\mathfrak{b}).$$

This is a closed immersion because the constructed functor

$$\Psi_{\breve{\mathbb{Q}}_p} : \mathrm{Rep}_{\breve{\mathbb{Q}}_p}(U(\mathfrak{b})) \to \mathrm{Isoc}(Z)$$

between Tannakian categories commutes with the $\breve{\mathbb{Q}}_p$-linear fibre functor obtained by restricting the objects to the closed point of Z (see Proposition 2.21.(b) of [25]). $\square$

7. Rigidity

7.1. **Introduction.** Let $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$ be a nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra such that the associated p-divisible group $\mathbb{X}(\mathfrak{a}^+)$ is connected. Assume that $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$ is equipped with a strongly non-trivial action of an algebraic group Q (see Definition 4.4.1) and let $\Gamma \subseteq Q(\mathbb{Q}_p)$ be a compact open subgroup preserving $\mathfrak{a}^+$. There is an induced morphism of Lie algebras $\mathrm{Lie}\, Q \to \mathrm{End}_F(\mathfrak{a})$ and after possibly shrinking Γ we may assume that there is a Lie bracket stable $\mathbb{Z}_p$-lattice $\Lambda \subset \mathrm{Lie}\, Q$ such that this maps Λ into $\mathrm{End}_F(\mathfrak{a}^+)$ (where $F = \varphi_{\mathfrak{a}^+}$). We assume as always that $p > n(\mathfrak{a})$.

Recall that for any F-stable Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{a}$ there is a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-subalgebra $\mathfrak{b}^+ \subseteq \mathfrak{a}^+$ defined by $\mathfrak{b}^+ = \mathfrak{b} \cap \mathfrak{a}^+$. In particular, such a $\mathfrak{b}$ defines a subspace

$$Z(\mathfrak{b}^+) \subseteq Z(\mathfrak{a}^+) = \frac{\tilde{\Pi}(\mathfrak{a})}{\Pi(\mathfrak{a}^+)}.$$

Theorem 7.1.1 (Rigidity)**.** *Let $Z \subseteq Z(\mathfrak{a}^+)$ be a Γ-stable integral closed formal subscheme. Then there is an F-stable Lie subalgebra $\mathfrak{b}_Z \subseteq \mathfrak{a}$ such that $Z = Z(\mathfrak{b}_Z^+)$.*

In other words, every subspace $Z \subseteq Z(\mathfrak{a}^+)$ that is stable under a strongly non-trivial action of a p-adic Lie group is strongly Tate-linear. This confirms an expectation of Chai–Oort, see Section 6.3 of [15]. Our proof is inspired by the proof of the rigidity result for biextensions by Chai–Oort in [18]. We will make a consistent effort throughout to point out the similarities between their proof and our proof, because our proof uses a different language than theirs (Diedonné–Lie algebras vs biextensions).

7.1.2. *A sketch of the proof.* If $\mathfrak{a}^+$ is abelian, then $Z(\mathfrak{a}^+)$ is a p-divisible formal group and the theorem was proved by Chai in [14]. The rough idea of our proof is to reduce to the abelian case by induction on the nilpotence degree, although in the actual proof we induct on the dimension of $Z(\mathfrak{a}^+)$. Let μ_1 be the smallest slope of $\mathfrak{a}$ and let $\mathfrak{b} \subset \mathfrak{a}$ be the maximal F-stable $\breve{\mathbb{Q}}_p$-subspace of $\mathfrak{a}$ that is isoclinic of that slope. Then Lemma 4.2.4 tells us $\mathfrak{b}$ lies in the center of $\mathfrak{a}$, and is therefore an F-stable Lie-algebra ideal of $\mathfrak{a}$. We will try to induct by passing from $\mathfrak{a}$ to $\mathfrak{a}/\mathfrak{b}$. For this, we would like to show that Z is stable under the action of the p-divisible formal group $Z(\mathfrak{b}^+) \subset Z(\mathfrak{a}^+)$.

Our actual argument is more complicated than that: We construct a p-divisible formal subgroup $Z(\mathfrak{b}_1^+) \subset Z(\mathfrak{b}^+)$ using the fact that the short exact sequence of F-isocrystals

$$0 \to \mathfrak{b} \to \mathfrak{a} \to \mathfrak{a}/\mathfrak{b} \to 0$$

splits uniquely and using results of Chai from [14] mentioned above. We then show in Proposition 7.4.2 that Z is stable under the action of $Z(\mathfrak{b}_1^+)$ and in Proposition 7.4.3 that $Z(\mathfrak{b}_1^+)$ is not the trivial subgroup. From there we can conclude by induction.

The core of the argument is the proof of Proposition 7.4.2, which is inspired by the proof of Proposition 7.3 of [18]. First we show that Z is stable under the action of $B \cdot Z(\mathfrak{b}_1^+)$, where B is an endomorphism of $Z(\mathfrak{b}_1^+)$ coming from the Lie algebra of the p-adic Lie group Γ. Then we use the nontriviality of the action of Γ to write down a linear combination of endomorphisms B that produce an isogeny of $Z(\mathfrak{b}_1^+)$, and the result follows since isogenies are surjective.

The proof that Z is stable under action of $B \cdot Z(\mathfrak{b}_1^+)$ goes via a kind of 'analysis of rigid functions' that is also found in Chai's proof of the abelian case in [14], although now we are working with perfected power series rings which makes everything more complicated. This part of the proof is more-or-less a direct translation of the proof of Proposition 7.3 of [18] to the setting of Dieudonné–Lie algebras, which we emphasise by following the notation of loc. cit. when we can.

7.2. **Preliminary reductions.** We start the proof by noticing that the statement is invariant under isogenies.

Lemma 7.2.1. *If $(\mathfrak{a}_1^+, \varphi_{\mathfrak{a}_1^+}, [-,-])$ and $(\mathfrak{a}_2^+, \varphi_{\mathfrak{a}_2^+}, [-,-])$ are isogenous Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras and Theorem 7.1.1 is true for $Z(\mathfrak{a}_1^+)$ for every choice of Γ, then the same is true for $Z(\mathfrak{a}_2^+)$.*

Proof. Suppose that $\mathfrak{a}_1^+ \to \mathfrak{a}_2^+$ is a morphism of Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras that is an isomorphism after inverting p. Then there is a closed immersion of Lie $\underline{\mathbb{Z}_p}$-algebras

$$T_p\mathbb{X}(\mathfrak{a}_1^+) \hookrightarrow T_p\mathbb{X}(\mathfrak{a}_2^+),$$

with finite flat cokernel. In particular, there is a closed immersion of group schemes

$$\Pi(\mathfrak{a}_1^+) \hookrightarrow \Pi(\mathfrak{a}_2^+)$$

that induces a finite flat surjective map (note the reversal of the order)

$$Z(\mathfrak{a}_2^+) \to Z(\mathfrak{a}_1^+). \tag{7.2.1}$$

For $i = 1, 2$, let Γ_i be a compact open subgroup of $Q(\mathbb{Q}_p)$ preserving $\mathfrak{a}_i^+$. Without loss of generality, we may assume that $\Gamma_1 \subseteq \Gamma_2$. Let $Z_2 \subseteq Z(\mathfrak{a}_2^+)$ be a Γ_2-stable integral closed formal subscheme as in the statement of Theorem 7.1.1. Note that the action of Γ_2 on Z_2 upgrades uniquely to an action of the affine group scheme $\underline{\Gamma_2}$ because this is true for $Z(\mathfrak{a}_2^+)$ by Lemma 4.4.3.

Let Z_1 be the scheme-theoretic image of Z_2 under (7.2.1). The formal subscheme $Z_1 \subseteq Z(\mathfrak{a}_1^+)$ is $\underline{\Gamma_1}$-stable by Lemma 2.1.4, and hence Γ_1-stable, and integral by construction. This means that the assumptions of Theorem 7.1.1 are also satisfied for Z_1. If we assume that Theorem 7.1.1 is true for $Z(\mathfrak{a}_1^+)$, then there is an F-stable Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{a} = \mathfrak{a}_1^+[\frac{1}{p}]$ with associated $\breve{\mathbb{Z}}_p$-lattice $\mathfrak{b}_1^+ \subseteq \mathfrak{a}_1^+$ such that

$$Z_1 = Z(\mathfrak{b}_1^+).$$

If we define $b_2^+ := a_2^+ \cap b$, we claim that $Z_2 = Z(\mathfrak{b}_2^+)$. Then both Z_2 and $Z(\mathfrak{b}_2^+)$ are closed subschemes of the inverse image of Z_1 under the finite map $Z(\mathfrak{a}_2^+) \to Z(\mathfrak{a}_1^+)$. Since this map is radicial and since Z_2 and $Z(\mathfrak{b}_2^+)$ are the formal spectra of complete Noetherian local domains of the same dimension as Z_1, it follows that $Z_2 = Z(\mathfrak{b}_2^+)$. $\square$

Hypothesis 7.2.2. Thanks to Lemma 7.2.1, during the proof of Theorem 7.1.1 throughout Section 7 we may (and will) assume that $\mathbb{X}(\mathfrak{b}^+)$ is completely slope divisible. In addition, we will assume that Z is not contained in $Z(\mathfrak{e}^+)$ for any F-stable Lie subalgebra $\mathfrak{e} \subsetneq \mathfrak{a}$. This can be done without loss of generality by replacing $\mathfrak{a}$ by the smallest F-stable Lie subalgebra $\mathfrak{d} \subset \mathfrak{a}$ such that $Z \subset Z(\mathfrak{d}^+)$.

7.3. **An approximation lemma.** Let μ_1 be the smallest slope of the Dieudonné–Lie algebra $\mathfrak{a}$ and let $\mathfrak{b} \subseteq \mathfrak{a}$ be the largest F-stable subspace that is isoclinic of slope μ_1. Lemma 4.2.4 tells us that $\mathfrak{b}$ is contained in the center of $\mathfrak{a}$ and therefore, if we equip it with the trivial Lie bracket, it becomes an F-stable Lie subalgebra of $\mathfrak{a}$. We then define $\mathfrak{c} := \frac{\mathfrak{a}}{\mathfrak{b}}$ and $\mathfrak{c}^+ := \frac{\mathfrak{a}^+}{\mathfrak{b}^+}$.

The short exact sequence

$$0 \to \mathfrak{b} \to \mathfrak{a} \to \mathfrak{c} \to 0$$

has a unique F-equivariant $\breve{\mathbb{Q}}_p$-linear splitting for slope reasons. Using Lemma 7.2.1 we may assume without loss of generality that this is induced by a (unique) F-equivariant $\breve{\mathbb{Z}}_p$-linear splitting of

$$0 \to \mathfrak{b}^+ \to \mathfrak{a}^+ \to \mathfrak{c}^+ \to 0.$$

We let $\sigma : \mathfrak{c}^+ \to \mathfrak{a}^+$ be the (unique) F-equivariant section of the natural map, and similarly we let $\rho : \mathfrak{a}^+ \to \mathfrak{b}^+$ be the (unique) F-equivariant retraction of the natural map. These maps are automatically equivariant for the action of Γ because Γ preserves slope filtrations[7], but they are generally *not* compatible with the Lie brackets. We also let ρ and σ denote the induced maps on p-divisible groups and their Tate-modules and universal covers.

Lemma 7.3.1. *Consider the morphism*

$$\beta : \tilde{\mathbb{X}}(\mathfrak{a}) \xrightarrow{\rho} \tilde{\mathbb{X}}(\mathfrak{b}) \longrightarrow \mathbb{X}(\mathfrak{b}^+).$$

Then the induced map

$$\Gamma(\mathbb{X}(\mathfrak{b}^+), \mathcal{O}) \to \Gamma(\tilde{\mathbb{X}}(\mathfrak{a}), \mathcal{O}) = \Gamma(Z(\mathfrak{a}^+), \mathcal{O})^{\mathrm{perf},\wedge}$$

factors through a complete restricted perfection of $\Gamma(Z(\mathfrak{a}^+), \mathcal{O})$. *See the end of the proof for a precise statement.*

Proof. Consider the Lie $\mathbb{Z}_p$-subalgebra

$$\tfrac{1}{p^n} T_p\mathbb{X}\left(\mathfrak{c}^+\right) \subseteq \tilde{\mathbb{X}}\left(\mathfrak{c}\right).$$

Using the BCH formula, we get a closed subgroup

$$\tfrac{1}{p^n}\Pi\left(\mathfrak{c}^+\right) \subseteq \tilde{\Pi}\left(\mathfrak{c}\right),$$

containing $\Pi\left(\mathfrak{c}^+\right)$. Therefore, it is stable under the translation action of $\Pi\left(\mathfrak{c}^+\right)$ and hence descends to a closed subscheme[8]

$$Z\left(\mathfrak{c}^+\right)[p^n] \subseteq Z\left(\mathfrak{c}^+\right).$$

We let $Z(\mathfrak{a}^+)_n$ denote the inverse image of $Z\left(\mathfrak{c}^+\right)[p^n]$ in $Z(\mathfrak{a}^+)$. Its inverse image under $\tilde{\Pi}(\mathfrak{a}) \to Z(\mathfrak{a}^+)$ can be described as the subgroup of $\tilde{\Pi}(\mathfrak{a})$ corresponding to the Lie $\mathbb{Z}_p$-subalgebra

$$\tilde{\mathbb{X}}(\mathfrak{b}) \oplus \tfrac{1}{p^n} T_p\mathbb{X}\left(\sigma\left(\mathfrak{c}^+\right)\right) \subseteq \tilde{\mathbb{X}}(\mathfrak{a})$$

[7] Indeed, if we conjugate our splitting by the action of an element of Γ, then we get another F-equivariant splitting which equals our original splitting by uniqueness

[8] If $\mathfrak{c}$ is an abelian Lie algebra then $Z(\mathfrak{c}^+)$ is a p-divisible group and the closed subscheme $Z(\mathfrak{c}^+)[p^n]$ defined above is indeed its p^n-torsion.

where σ is the unique F-equivariant $\breve{\mathbb{Z}}_p$-linear section of $\mathfrak{a}^+ \to \mathfrak{c}^+$ as before. We will often omit the σ by abuse of notation because it is unique.

Claim 7.3.2. *There exists a morphism $\eta_n : Z(\mathfrak{a}^+)_n \to \mathbb{X}(\mathfrak{b}^+)$ such that the following diagram commutes*

$$\begin{tikzcd} \mathbb{X}(\mathfrak{b}^+) \arrow[r, hook] \arrow[d, "p^n"] & Z(\mathfrak{a}^+)_n \arrow[dl, dotted, "\eta_n"] \\ \mathbb{X}(\mathfrak{b}^+). & \end{tikzcd}$$

Proof. Consider the diagram

$$\begin{tikzcd} \tilde{\mathbb{X}}(\mathfrak{b}) \arrow[r, hook] \arrow[d] & \tilde{\mathbb{X}}(\mathfrak{b}) \oplus \frac{1}{p^n} T_p\mathbb{X}(\mathfrak{c}^+) \arrow[l, bend right, "\rho"'] \arrow[d] \\ \mathbb{X}(\mathfrak{b}^+) \arrow[r, hook] & Z(\mathfrak{a}^+)_n. \end{tikzcd}$$

To prove the claim, it suffices to prove that the map given by ρ, followed by the natural projection $\tilde{\mathbb{X}}(\mathfrak{b}) \to \mathbb{X}(\mathfrak{b}^+)$ and by the multiplication by p^n is invariant under the action of $\Pi(\mathfrak{a}^+)$ on the source; this action is given by multiplication inside the group $\tilde{\Pi}(\mathfrak{a})$ as governed by the BCH formula.

Let (v, w) be an R-point of $\tilde{\mathbb{X}}(\mathfrak{b}) \oplus \frac{1}{p^n} T_p\mathbb{X}(\mathfrak{c}^+)$ and let $(v', w') \in \Pi(\mathfrak{a}^+)$. Using the assumption that $p > n(\mathfrak{a})$ and [38], the BCH formula tells us that

$$\rho\left((v', w') \cdot (v, w)\right) = v + v' + \rho\left(\mathbb{Z}_{(p)}\text{-linear combinations of iterated Lie brackets of } w \text{ and } w'\right).$$

Since iterated Lie brackets of w and w' (and $\mathbb{Z}_{(p)}$-linear combinations of them) are all in

$$\tfrac{1}{p^n} T_p\mathbb{X}\left(\mathfrak{a}^+\right)(R)$$

by the integrality of the Lie-bracket, and since ρ sends $T_p\mathbb{X}(\mathfrak{a}^+)$ to $T_p\mathbb{X}(\mathfrak{b}^+)$, we see that the difference between $\rho\left((v', w') \cdot (v, w)\right)$ and v lands in

$$\tfrac{1}{p^n} T_p\mathbb{X}(\mathfrak{b}^+)(R).$$

This is precisely the kernel of

$$\tilde{\mathbb{X}}(\mathfrak{b})(R) \longrightarrow \mathbb{X}(\mathfrak{b}^+)(R) \xrightarrow{p^n} \mathbb{X}(\mathfrak{b}^+)(R)$$

and so we are done. □

We define now $Z(\mathfrak{a}^+)[F^n] \subseteq Z(\mathfrak{a}^+)$ to be closed subscheme defined by $\varphi_{\mathrm{rel}}^n(\mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of the coordinate ring of $Z(\mathfrak{a}^+)$ and where φ_{rel} is the Frobenius relative to $\overline{\mathbb{F}}_p$ on the coordinate ring. We will use this notation later also for other formal spectra of complete local Noetherian rings over $\overline{\mathbb{F}}_p$ with residue field $\overline{\mathbb{F}}_p$.

Let a and r be sufficiently divisible positive integers such that $a/r = \mu_1$ is the slope of $\mathbb{X}(\mathfrak{b}^+)$. Since $\mathbb{X}(\mathfrak{b}^+)$ is completely slope divisible this means that

$$\mathbb{X}(\mathfrak{b}^+)[p^{na}] = \mathbb{X}(\mathfrak{b}^+)[F^{nr}].$$

Let μ_0 be a rational number that is smaller than μ_1 but larger than all the slopes of the Dieudonné–Lie algebra $\mathfrak{a}/\mathfrak{b}$. We can write $\mu_0 = a/s$ for $s > r$, possibly after multiplying both a and r by suitable positive integers.

Claim 7.3.3. *There is an integer n_0 such that for all $n \geq n_0$ we have $Z(\mathfrak{a}^+)[F^{ns}] \subseteq Z(\mathfrak{a}^+)_{na}$. Moreover, the restriction of η_{na} to $Z(\mathfrak{a}^+)[F^{ns}]$ factors through $\mathbb{X}(\mathfrak{b}^+)[F^{ns}]$.*

Proof. For the first claim it suffices to show that there exists an n_0 such that $Z(\mathfrak{c}^+)[F^{ns}] \subseteq Z(\mathfrak{c}^+)[p^{na}]$ for all $n \geq n_0$. If the Lie algebra $\mathfrak{c}$ is abelian, then $Z(\mathfrak{c}^+)[p^n]$ is the actual p^n-torsion of the p-divisible group $\mathbb{X}(\mathfrak{c}^+)$ and the result follows from general properties of slopes. In general the result is obtained by induction because $Z(\mathfrak{c}^+)[p^n]$ is a tower of torsors for $H_i[p^n]$ where H_i is a p-divisible group of slope strictly greater than μ_0. The second part of the claim follows from the fact that maps between formal spectra of complete local rings induce local homomorphisms on coordinate rings. $\square$

Now write $\mathbb{X}(\mathfrak{b}^+) = \operatorname{Spf} \overline{\mathbb{F}}_p[[u_1, \cdots, u_d]]$ with

$$[p^a]^* u_j = u_j^{p^r}.$$

This is possible because $\mathbb{X}(\mathfrak{b}^+)$ is completely slope divisible. The map $\beta : \tilde{\mathbb{X}}(\mathfrak{a}) \to \mathbb{X}(\mathfrak{b}^+)$ corresponds to a d-tuple of topologically nilpotent elements $a_1, \cdots, a_d$ in

$$\Gamma(\tilde{\mathbb{X}}(\mathfrak{a}), \mathcal{O}) = \Gamma(Z(\mathfrak{a}^+), \mathcal{O})^{\mathrm{perf},\wedge}.$$

The geometric discussion above tells us that the image of $a_j^{p^{nr}}$ in

$$\Gamma\left(\tilde{\mathbb{X}}(\mathfrak{b}) \oplus \tfrac{1}{p^{na}} T_p \mathbb{X}\left(\mathfrak{c}^+\right), \mathcal{O}\right)$$

lies in the image of

$$R = \Gamma(Z(\mathfrak{a}^+)_{na}, \mathcal{O}) \to \Gamma\left(\tilde{\mathbb{X}}(\mathfrak{b}) \oplus \tfrac{1}{p^{na}} T_p \mathbb{X}\left(\mathfrak{c}^+\right), \mathcal{O}\right)$$

Since $Z(\mathfrak{a}^+)_{na} \supseteq Z(\mathfrak{a}^+)[F^{ns}]$ this implies that for all $n \geq n_0$ the elements $a_i^{p^{nr}}$ are congruent to elements $c_{i,n}$ of R modulo $\varphi_{\mathrm{rel}}^{ns}(\mathfrak{m}_R)$ for all $n \geq n_0$. By definition, see Section A.2, this means that for all i we have

$$a_i \in \Gamma(Z(\mathfrak{a}^+), \mathcal{O})^{\mathrm{perf},\wedge}_{r,s,\geq n_0}.$$

$\square$

Remark 7.3.4. In [18] Chai and Oort construct the morphisms η_n for all n by explicitly working with biextensions. They then define $a_{n,j} := \eta_{na}^* u_j \in R/\varphi_{\mathrm{rel}}^{ns}(\mathfrak{m}_R)$ which satisfy $a_{n,j}^{p^r} = a_{n+1,j}$ in $R/\varphi_{\mathrm{rel}}^{ns+r}(\mathfrak{m}_R)$, and use this to motivate the definitions of certain complete restricted perfections. They then use these complete restricted perfections to define (a variant of) ρ. In our proof, we are doing the opposite. We first constructed ρ and then used it to define the morphisms η_n.

7.4. **Proof of the Theorem 7.1.1.** Let the notation be as in the statement of Theorem 7.1.1. We want to prove the theorem by induction on the dimension of $Z(\mathfrak{a}^+)$. The base case when $Z(\mathfrak{a}^+)$ has dimension zero is vacuous because then $Z(\mathfrak{a}^+)$ is a point. We will let $\mathfrak{b}$ be as in Section 7.3. As in Hypothesis 7.2.2, we will assume that $\mathbb{X}(\mathfrak{b}^+)$ is completely slope divisible, and that Z is not contained in $Z(\mathfrak{e}^+)$ for any F-stable Lie subalgebra $\mathfrak{e} \subsetneq \mathfrak{a}$.

Let $\tilde{Z} \hookrightarrow \tilde{\Pi}(\mathfrak{a}) = \tilde{\mathbb{X}}(\mathfrak{a})$ be the perfection of the closed immersion $Z \hookrightarrow Z(\mathfrak{a}^+)$ and consider the Γ-equivariant morphism

$$\alpha : \tilde{Z} \longhookrightarrow \tilde{\mathbb{X}}(\mathfrak{a}) \xrightarrow{\rho} \tilde{\mathbb{X}}(\mathfrak{b}) \longrightarrow \mathbb{X}(\mathfrak{b}^+) \tag{7.4.1}$$

We denote by $T \subseteq \mathbb{X}(\mathfrak{b}^+)$ the scheme-theoretic image of α.

Lemma 7.4.1. *There is an F-stable $\breve{\mathbb{Q}}_p$-submodule $\mathfrak{b}_1 \subseteq \mathfrak{b}$ such that $T = Z(\mathfrak{b}_1^+)$. Moreover, the morphism*

$$\tilde{\alpha} : \tilde{Z} \hookrightarrow \tilde{\mathbb{X}}(\mathfrak{a}) \xrightarrow{\rho} \tilde{\mathbb{X}}(\mathfrak{b})$$

factors through $\tilde{\mathbb{X}}(\mathfrak{b}_1)$.

Proof. The action of the profinite group Γ on all the objects in (7.4.1) upgrades to an action of the affine scheme $\underline{\Gamma}$ by Lemma 4.4.3. Therefore the formal scheme $T \subseteq \mathbb{X}(\mathfrak{b}^+)$ is integral and $\underline{\Gamma}$-stable by Lemma 2.1.4, and in particular Γ-stable. Because the action of Γ on $\mathfrak{a}^+$ is strongly nontrivial, the induced action on $\mathfrak{b}^+$ is also strongly nontrivial. Now the main result of [14] tells us that T is a p-divisible subgroup. In other words, $T = Z(\mathfrak{b}_1^+)$ for some F-stable $\breve{\mathbb{Q}}_p$-submodule $\mathfrak{b}_1 \subseteq \mathfrak{b}$.

The second assertion of the lemma is that the dotted arrow in the following diagram exists

$$\begin{tikzcd}
\tilde{Z} \arrow[r, hook] \arrow[drr, dotted] \arrow[drrr, bend right] & \tilde{\mathbb{X}}(\mathfrak{a}) \arrow[r, "\rho"] & \tilde{\mathbb{X}}(\mathfrak{b}) \arrow[r] & \mathbb{X}(\mathfrak{b}^+) \\
 & & \tilde{\mathbb{X}}(\mathfrak{b}_1) \arrow[r] \arrow[u, hook] & \mathbb{X}(\mathfrak{b}_1^+). \arrow[u, hook]
\end{tikzcd}$$

This follows from the universal property of the perfection of formal schemes: Indeed $\tilde{\mathbb{X}}(\mathfrak{b}_1) \to \mathbb{X}(\mathfrak{b}_1^+)$ is the perfection of the target and $\tilde{Z}$ is perfect, and thus the dotted arrow exists uniquely. ☐

The rest of the proof of Theorem 7.1.1 will rely on the following two results.

Proposition 7.4.2. *The closed formal subscheme $Z \subseteq Z(\mathfrak{a}^+)$ is stable under the action of $Z(\mathfrak{b}_1^+)$.*

Proposition 7.4.3. *The subgroup $Z(\mathfrak{b}_1^+) \subseteq Z(\mathfrak{b}^+)$ is not the trivial subgroup.*

Proof of Theorem 7.1.1. Recall that we have assumed without loss of generality that there is no F-stable Lie-subalgebra $\mathfrak{c} \subsetneq \mathfrak{a}$ such that $Z \subset Z(\mathfrak{c}^+)$. We are then trying to prove that $Z = Z(\mathfrak{a}^+)$.

It follows from Proposition 7.4.2 and Proposition 7.4.3 that there is an inclusion

$$\frac{Z}{Z(\mathfrak{b}_1^+)} \subseteq \frac{Z(\mathfrak{a}^+)}{Z(\mathfrak{b}_1^+)} = Z\left(\frac{\mathfrak{a}^+}{\mathfrak{b}_1^+}\right)$$

and that the dimension of $Z\left(\frac{\mathfrak{a}^+}{\mathfrak{b}_1^+}\right)$ is smaller than the dimension of $Z(\mathfrak{a}^+)$. Therefore, by the inductive hypothesis, we find that

$$\frac{Z}{Z(\mathfrak{b}_1^+)} = Z(\mathfrak{d}^+)$$

for some F-stable Lie subalgebra $\mathfrak{d} \subseteq \frac{\mathfrak{a}}{\mathfrak{b}_1}$. However, if $\mathfrak{d} \neq \frac{\mathfrak{a}}{\mathfrak{b}_1}$ then Z is contained in $Z(\pi^{-1}(\mathfrak{d}^+))$, where

$$\pi : \mathfrak{a} \to \frac{\mathfrak{a}}{\mathfrak{b}_1}$$

is the natural projection. This contradicts our assumption that Z is not contained in $Z(\mathfrak{e}^+)$ for any F-stable Lie subalgebra $\mathfrak{e} \subsetneq \mathfrak{a}$. Thus

$$\frac{Z}{Z(\mathfrak{b}_1^+)} = \frac{Z(\mathfrak{a}^+)}{Z(\mathfrak{b}_1^+)}$$

and we find that $Z = Z(\mathfrak{a}^+)$. ☐

We end Section 7 with a proof of Proposition 7.4.2 and Proposition 7.4.3. Note that Proposition 7.4.2 is an analogue of Proposition 7.3 of [18] and Proposition 7.4.3 is a generalisation of Proposition 6.2 of [*ibid.*], as explained in Remark 7.6.2.

7.5. **Proof of Proposition 7.4.2.** Let $X \in \Lambda$ and choose $n \gg 0$ such that $g_n = \exp(p^n X)$ converges to an element of Γ. By the discussion in the beginning of Section 7 and the discussion before Lemma 4.4.3 in Section 4.4, the element X induces an automorphism of $\tilde{\Pi}(\mathfrak{a})$ preserving $\Pi(\mathfrak{a}^+)$ and thus an endomorphism of the formal scheme $Z(\mathfrak{a}^+)$.

We will write B for the induced endomorphism of $\mathfrak{b}^+$ and C for the induced endomorphism of $\mathfrak{c}^+$. We will consider the following commutative diagram throughout this section (if we take global sections of all the formal schemes involved in the diagram, then we recover a generalisation of the diagram on page 64 of [18])

(7.5.1)
$$\begin{tikzcd}
Z \arrow[r, hook] & Z(\mathfrak{a}^+) \arrow[r, equal] & Z(\mathfrak{a}^+) & \\
& \mathbb{X}(\mathfrak{b}_1^+) \times Z \arrow[r] \arrow[u, "\star"] & \mathbb{X}(\mathfrak{b}^+) \times Z(\mathfrak{a}^+) \arrow[u, "\star"] & \\
& \mathbb{X}(\mathfrak{b}_1^+) \times Z \arrow[r, hook] \arrow[u, "B\times 1"] \arrow[uul, dotted, bend left] & \mathbb{X}(\mathfrak{b}^+) \times Z(\mathfrak{a}^+) \arrow[u, "B\times 1"] & \tilde{\Pi}(\mathfrak{a}) \arrow[uul] \\
& \tilde{\mathbb{X}}(\mathfrak{b}_1) \times \tilde{Z} \arrow[r, hook] \arrow[u] & \tilde{\mathbb{X}}(\mathfrak{b}) \times \tilde{\Pi}(\mathfrak{a}) \arrow[r, "B\times 1"] \arrow[u] & \tilde{\mathbb{X}}(\mathfrak{b}) \times \tilde{\Pi}(\mathfrak{a}) \arrow[u, "\star"] \\
& \tilde{Z} \times \tilde{Z} \arrow[r, hook] \arrow[u, "\alpha\times 1"] & \tilde{\Pi}(\mathfrak{a}) \times \tilde{\Pi}(\mathfrak{a}) \arrow[u, "\rho\times 1"] & \\
& \tilde{Z} \arrow[r] \arrow[u, "(\varphi_{\mathrm{rel}}^{nr}, 1)"] & \tilde{\Pi}(\mathfrak{a}). \arrow[u, "(\varphi_{\mathrm{rel}}^{nr}, 1)"] &
\end{tikzcd}$$

Let us elaborate on the maps appearing in the diagram:

- There is an action of $\tilde{\Pi}(\mathfrak{b}) = \tilde{\mathbb{X}}(\mathfrak{b})$ on $\tilde{\Pi}(\mathfrak{a})$ by left translation (or right translation, since it is central by Lemma 4.2.4), which we denote by $\star$. This induces an action of $\mathbb{X}(\mathfrak{b}^+)$ on $Z(\mathfrak{a}^+)$, which we also denote by $\star$. Its restriction to $\mathbb{X}(\mathfrak{b}_1^+)$ is also denoted by $\star$.
- In various places we use the natural maps $\tilde{\mathbb{X}}(\mathfrak{b}) \to \mathbb{X}(\mathfrak{b}^+), \tilde{\mathbb{X}}(\mathfrak{b}_1) \to \mathbb{X}(\mathfrak{b}_1^+), \tilde{\Pi}(\mathfrak{a}) \to Z(\mathfrak{a}^+)$ without naming them.

An important step in the proof will be to prove the following result

Proposition 7.5.1. *The dotted arrow in* (7.5.1) *exists.*

We start by studying the automorphism of $\frac{1}{p^n} T_p \mathbb{X}(\mathfrak{c}^+)$ induced by $g_n = \mathrm{Exp}(p^n X)$ for $n \gg 0$. It follows from the Taylor series for the exponential map that g_n is congruent to the identity modulo $T_p \mathbb{X}(\mathfrak{c}^+)$ and thus g_n induces the identity on $Z(\mathfrak{c}^+)[p^n]$. Similarly we can write the automorphism of $\tilde{\mathbb{X}}(\mathfrak{b})$ induced by g_n as

$$1 + p^n \circ B + \sum_{j \geq 2} \frac{p^{jn}}{j!} B^j.$$

We see that the automorphism of $\tilde{\mathbb{X}}(\mathfrak{b}) \oplus \frac{1}{p^n} T_p\mathbb{X}(\mathfrak{c}^+)$ induced by g_n is congruent modulo $T_p\mathbb{X}(\mathfrak{b}^+) \oplus T_p\mathbb{X}(\mathfrak{c}^+)$ to

$$x \mapsto \left(B \circ [p^n] \circ \rho(x) + \sum_{j \geq 2} \frac{p^{jn}}{j!} B^j \circ \rho(x) \right) \star x,$$

where $\star$ denotes the translation action of $\tilde{\mathbb{X}}(\mathfrak{b})$ on $\tilde{\mathbb{X}}(\mathfrak{b}) \oplus \frac{1}{p^n} T_p\mathbb{X}(\mathfrak{c}^+)$.[9]

We know that g_n induces an automorphism of the quotient $Z(\mathfrak{a}^+)_n$ and there g_n acts via the following formula (where we note that $p^n \circ \rho(x) =: \eta_n(x)$)

$$g_n(x) = \left(B \circ \eta_n(x) + \sum_{j \geq 2} \frac{p^{(j-1)n}}{j!} B^j \circ \eta_n(x) \right) \star x,$$

where $\star$ denotes the translation action of $\mathbb{X}(\mathfrak{b}^+)$ on $Z(\mathfrak{a}^+)_n$, and where B denotes the endomorphism of $\mathbb{X}(\mathfrak{a}^+)$ induced by B. If we now look at the action of $\exp(p^{na}X)$ on $Z(\mathfrak{a}^+)_{na}$ then it is given by the following formula (where 1 denotes the identity map on $Z(\mathfrak{a}^+)_{na}$)

$$g_{na} = \left(B \circ \eta_{na} + \sum_{j \geq 2} \frac{p^{(j-1)na}}{j!} B^j \circ \eta_{na} \right) \star 1$$

Claim 7.5.2. *There exists a constant c such that for all sufficiently large n there is an equality of endomorphisms*

$$\left(B \circ \eta_{na} + \sum_{j \geq 2} \frac{p^{(j-1)na}}{j!} B^j \circ \eta_{na} \right) \star 1 = (B \circ \eta_{na}) \star 1$$

of the scheme $Z(\mathfrak{a}^+)_{na}[F^{2nr-c}]$.

Proof. This just comes down to showing that

$$\frac{p^{(j-1)na}}{j!} B^j \circ \eta_{na} \equiv 0 \mod \varphi_{\mathrm{rel}}^{2nr-c}(\mathfrak{m}).$$

It is not so hard to see that η_{na} itself is zero modulo $\varphi_{\mathrm{rel}}^{nr}(\mathfrak{m})$ because $p^{na} = \varphi_{\mathrm{rel}}^{nr}$ on $\mathbb{X}(\mathfrak{b}^+)$. Thus we just need to remark that the p-adic valuation of

$$\frac{p^{(j-1)na}}{j!}$$

is at least $na - c$ for $n \gg 0$ to conclude, which is an elementary calculation. □

Proof of Proposition 7.5.1. Let $E \in R = \Gamma(Z(\mathfrak{a}^+), \mathcal{O})$ be a function that vanishes on Z. To prove Proposition 7.5.1, we have to show that the pullback

$$(B \times 1)^* \circ \star^* E$$

vanishes. We first note that the map $\tilde{Z} \times Z \to \mathbb{X}(\mathfrak{b}_1^+) \times Z$ given by composing $\alpha \times 1$ with the natural map $\tilde{\mathbb{X}}(\mathfrak{b}) \to \mathbb{X}(\mathfrak{b}_1^+)$ is injective on coordinate rings. Indeed this is true for $\tilde{\mathbb{X}}(\mathfrak{b}) \to \mathbb{X}(\mathfrak{b}_1^+)$ by Lemma A.1.6, it is true for $\alpha : \tilde{Z} \to \mathbb{X}(\mathfrak{b}_1)$ by the definition of scheme-theoretic image and

[9] The same formula appears in Proposition 5.3.2 of [18] and inspired ours.

by Lemma 2.1.6, the induced map on completed tensor products is also injective. Thanks to this observation, it suffices to show that the further pullback

$$(\alpha \times 1)^* \circ (B \times 1)^* \circ \star^* E \tag{7.5.2}$$

vanishes. By Proposition 2.1 of [14], there is an injective local homomorphism

$$\Gamma(Z, \mathcal{O}) \to \overline{\mathbb{F}}_p[[X_1, \cdots, X_m]] =: S.$$

If we write W for $\operatorname{Spf} S$, we have the following commutative diagram

$$\begin{tikzcd} \tilde{W} \times \tilde{W} \arrow[r] & \tilde{Z} \times \tilde{Z} \\ \tilde{W} \arrow[u, "(\varphi_{\mathrm{rel}}^{nr}, 1)"] \arrow[r] & \tilde{Z}. \arrow[u, "(\varphi_{\mathrm{rel}}^{nr}, 1)"'] \end{tikzcd}$$

Note that the induced map

$$\Gamma(\tilde{Z} \times \tilde{Z}, \mathcal{O}) \to \Gamma(\tilde{W} \times \tilde{W}, \mathcal{O})$$

is again injective because the formation of completed perfections commutes with completed tensor products by Lemma A.1.12 and completed perfections of injective maps are injective by Lemma A.1.9. Thus we want to prove that the image of (7.5.2) in

$$\Gamma(\tilde{W} \times \tilde{W}, \mathcal{O}) = \overline{\mathbb{F}}_p[[X_1, \cdots, X_m, Y_1, \cdots, Y_m]]^{\mathrm{perf}, \wedge}$$

is zero. Choose coordinates $u_1, \cdots u_d$ for $\mathbb{X}(\mathfrak{b})$ and coordinates $v_1, \cdots, v_e$ for $Z(\mathfrak{a}^+)$. The pullbacks of $u_1, \cdots, u_d$ under

$$\tilde{W} \longrightarrow \tilde{Z} \xrightarrow{\alpha} \mathbb{X}(\mathfrak{b})$$

are topologically nilpotent elements $g_1, \cdots, g_d$ contained, by Lemma 7.3.1, in $\overline{\mathbb{F}}_p[[X_1, \cdots, X_m]]^{\mathrm{perf}, \wedge}_{r,s, \geq n_0}$. Similarly the pullbacks of $v_1, \cdots, v_e$ under

$$\tilde{W} \longrightarrow \tilde{Z} \longrightarrow \tilde{\Pi}(\mathfrak{a}) \longrightarrow Z(\mathfrak{a}^+)$$

are given by topologically nilpotent elements

$$h_1, \cdots, h_e \in \overline{\mathbb{F}}_p[[X_1, \cdots, X_m]].$$

If we write the pullback of a to $\mathbb{X}(\mathfrak{b}^+) \times Z(\mathfrak{a}^+)$ as a power series $f(u_1, \cdots, u_d, v_1, \cdots, v_e)$, then the image of (7.5.2) in

$$\overline{\mathbb{F}}_p[[X_1, \cdots, X_m, Y_1, \cdots, Y_m]]^{\mathrm{perf}, \wedge}$$

is given by

$$f(g_1, \cdots, g_d, h_1, \cdots, h_e).$$

We note that in this case the composition of completed perfected power series makes sense. Proposition 7.5.1 follows then from the following claim. ☐

Claim 7.5.3. *The function $f(g_1, \cdots, g_d, h_1, \cdots, h_e)$ vanishes.*

Proof. Proposition 6.4 of [18] (stated in Appendix A as Theorem A.2.4) tells us that to show that $f(g_1, \cdots, g_d, h_1, \cdots, h_e) = 0$ it suffices to find an integer r and a sequence of integers d_n such that

$$\lim_{n \to \infty} \frac{p^{rn}}{d_n} = 0$$

and such that (with $q = p^r$)

$$f(g_1(\underline{X})^{q^n}, \cdots, g_d(\underline{X})^{q^n}, h_1(\underline{X}), \cdots, h_e(\underline{X})) \equiv 0 \mod (X_1, \cdots, X_m)^{d^n}.$$

Note that

$$f(g_1(\underline{X})^{q^n}, \cdots, g_d(\underline{X})^{q^n}, h_1(\underline{X}), \cdots, h_e(\underline{X}))$$

is precisely the pullback of $f(g_1, \cdots, g_d, h_1, \cdots, h_e)$ under $(\varphi_{\mathrm{rel}}^{nr}, 1)$. We will apply this theorem with r as above, where we recall that $a/r = \mu_1$ is the slope of $\mathbb{X}(\mathfrak{b}_1^+)$, and we will take $d^n = p^{2nrm-c'}$ where c' is some constant. Proving the congruence comes down to chasing the diagram in (7.5.1) and applying the congruence from Claim 7.5.2.

Indeed, our assumption that Z is stable under the action of Γ tells us that the pullback of E under the automorphism induced by $g_{na} = \exp(p^{na}X)$ of $Z(\mathfrak{a}^+)$ maps to zero in $\Gamma(Z, \mathcal{O})$, and so the further pullback to $\Gamma(\tilde{\Pi}(\mathfrak{a}), \mathcal{O})$ maps to zero in $\Gamma(\tilde{Z}, \mathcal{O})$. The congruence

$$\exp(p^n X) \equiv (B \circ \eta_{na}) \star 1 \mod \varphi_{\mathrm{rel}}^{2nr-c}(\mathfrak{m})$$

of Claim 7.5.2 tells us that the pullback of E via $(B \circ p^{na} \circ \rho) \star 1$, maps to zero in $\Gamma(Z, \mathcal{O})$ modulo $\varphi_{\mathrm{rel}}^{2nr-c}(\mathfrak{m})$. Since $\eta_{na} = [p^n] \circ \rho_{na}$ the further pullback to $\Gamma(\tilde{Z}(\mathfrak{a}), \mathcal{O})$ is the same as the pullback of E via

$$\tilde{Z} \longrightarrow \tilde{\Pi}(\mathfrak{a}) \xrightarrow{(\rho,1)} \tilde{\mathbb{X}}(\mathfrak{b}) \times \tilde{\Pi}(\mathfrak{a}) \xrightarrow{(B\circ p^{na},1)} \tilde{\mathbb{X}}(\mathfrak{b}) \times \tilde{\Pi}(\mathfrak{a}) \xrightarrow{\star} \tilde{\Pi}(\mathfrak{a}) \longrightarrow Z(\mathfrak{a}^+).$$

Since $p^{na} = \varphi_{\mathrm{rel}}^{nr}$ on $\mathbb{X}(\mathfrak{b}^+)$ we can identify this morphism instead with the composition of $\tilde{Z} \to \tilde{\Pi}(\mathfrak{a})$ of

$$\tilde{\Pi}(\mathfrak{a}) \xrightarrow{(\varphi_{\mathrm{rel}}^{nr},1)} \tilde{\Pi}(\mathfrak{a}) \times \tilde{\Pi}(\mathfrak{a}) \xrightarrow{\rho\times 1} \tilde{\mathbb{X}}(\mathfrak{b}) \times \tilde{\Pi}(\mathfrak{a}) \xrightarrow{(B,1)} \tilde{\mathbb{X}}(\mathfrak{b}) \times \tilde{\Pi}(\mathfrak{a}) \xrightarrow{\star} \tilde{\Pi}(\mathfrak{a}) \longrightarrow Z(\mathfrak{a}^+).$$

To summarise, the pullback of E along the composition of this morphism with $\tilde{Z} \to \tilde{\Pi}(\mathfrak{a})$ vanishes modulo $\varphi_{\mathrm{rel}}^{2nr-c}$. By the commutativity of the big diagram (7.5.1) this is precisely saying that the pullback of E along the five vertical maps in the second column vanishes modulo $\varphi_{\mathrm{rel}}^{2nr-c}(\mathfrak{m})$, and so the same is true for $f(g_1(\underline{X})^{q^n}, \cdots, g_d(\underline{X})^{q^n}, h_1(\underline{X}), \cdots, h_e(\underline{X}))$. To be precise, it is zero modulo

$$(X_1^{p^{2nr-c}}, \cdots, X_m^{p^{2nr-c}}) \supseteq (X_1, \cdots, X_n)^{p^{2mnr-cm}},$$

and so Proposition 6.4 of [18] applies and we are done. □

Proof of Proposition 7.4.2. Proposition 7.4.2 follows from Proposition 7.5.1 by applying Lemma 4.1.1 of [14], as in the proof of Theorem 7.2 of [18] or the proof of Theorem 4.3 of [14]. The assumptions of this lemma are satisfied because the action of Γ is strongly non-trivial. □

7.6. **Proof of Proposition 7.4.3.** Suppose that the scheme-theoretic image of α is trivial and let σ be the (unique) F-equivariant section of $\mathfrak{a} \to \mathfrak{c}$. Our assumption implies that $\tilde{Z}$ is contained in the image of this section. In other words, it is contained in

$$\tilde{\mathbb{X}}(\sigma(\mathfrak{c})) \subseteq \tilde{\mathbb{X}}(\mathfrak{a}) = \tilde{\Pi}(\mathfrak{a}).$$

We are going to show that $\tilde{\mathbb{X}}(\sigma(\mathfrak{c}))$ is a Lie subalgebra of $\tilde{\mathbb{X}}(\mathfrak{a})$, contradicting our assumption that Z is not contained in $Z(\mathfrak{d}^+)$ for an F-stable Lie-subalgebra $\mathfrak{d} \subseteq \mathfrak{a}$. This contradiction uses the following claim.

Claim 7.6.1. *The subspace*

$$\tilde{\mathbb{X}}(\sigma(\mathfrak{c})) \subseteq \tilde{\mathbb{X}}(\mathfrak{a})$$

is stable under the Lie bracket if and only if the F-stable $\breve{\mathbb{Q}}_p$-subspace

$$\sigma(\mathfrak{c}) \subseteq \mathfrak{a}$$

is stable under the Lie bracket.

Proof. This is an immediate consequence of Lemma 4.2.6. □

Let Z' be the inverse image of Z in $\tilde{\Pi}(\mathfrak{a})$, and note that

$$Z' \to Z$$

is a torsor for $\Pi(\mathfrak{a}^+)$. The restriction of ρ to Z' factors through $T_p\mathbb{X}(\mathfrak{b}^+) \subseteq \tilde{\mathbb{X}}(\mathfrak{b})$ by the assumption that the scheme-theoretic image of $\tilde{Z}$ under α is trivial. In fact Z' surjects onto $T_p\mathbb{X}(\mathfrak{b}^+)$ because ρ is equivariant for the translation action of $T_p\mathbb{X}(\mathfrak{b}^+)$ via $T_p\mathbb{X}(\mathfrak{b}^+) \subseteq \Pi(\mathfrak{a}^+)$. Moreover since ρ is a retraction of the inclusion of $\tilde{\mathbb{X}}(\mathfrak{b})$ it follows that

$$\rho_R : Z'(R) \to T_p\mathbb{X}(\mathfrak{b}^+)(R)$$

is surjective for every $\overline{\mathbb{F}}_p$-algebra R.

Let $x = (v, w) \in Z'(R)$ with v denoting $\rho(x)$ and w denoting the image of x in $\tilde{\Pi}(\mathfrak{c})$ and similarly let $x' = (v', w') \in \Pi(\mathfrak{a}^+)$. Then the action of x' on x can be described as

$$(v + v' + \rho([w, w']), w \cdot w').$$

where $\cdot$ denotes the multiplication in $\tilde{\Pi}(\mathfrak{c})$. Since

$$\rho(Z'(R)) = T_p\mathbb{X}(\mathfrak{b}^+)(R)$$

is a subgroup containing v, v' and $v + v' + \rho([w, w'])$ we see that it must contain $\rho([w, w'])$. The existence of the section σ shows that

$$Z'(R) \to \tilde{\Pi}(\mathfrak{c})(R)$$

is surjective for all R. So we can choose w arbitrarily and similarly we can choose w' arbitrarily. In other words

$$\rho(Z'(R)) = T_p\mathbb{X}(\mathfrak{b}^+)(R)$$

contains the image of the map

$$T_p\mathbb{X}\left(\sigma\left(\mathfrak{c}^+\right)\right)(R) \times \tilde{\mathbb{X}}(\sigma(\mathfrak{c}))(R) \to \tilde{\mathbb{X}}(\mathfrak{a})(R) \to \tilde{\mathbb{X}}(\mathfrak{b})(R), \tag{7.6.1}$$

where the last map is given by ρ and the first by the Lie bracket. Since the pairing is bilinear and $\tilde{\mathbb{X}}(\sigma(\mathfrak{c}))(R)$ is a $\mathbb{Q}_p$-vector space we see that the image of (7.6.1) is equal to the image of

$$\tilde{\mathbb{X}}(\sigma(\mathfrak{c}))(R) \times \tilde{\mathbb{X}}(\sigma(\mathfrak{c}))(R) \to \tilde{\mathbb{X}}(\mathfrak{a})(R) \to \tilde{\mathbb{X}}(\mathfrak{b})(R).$$

But this means that the morphism

$$\tilde{\mathbb{X}}(\sigma(\mathfrak{c})) \times \tilde{\mathbb{X}}(\sigma(\mathfrak{c})) \to \tilde{\mathbb{X}}(\mathfrak{a}) \to \tilde{\mathbb{X}}(\mathfrak{b}).$$

factors through the inclusion

$$T_p\mathbb{X}\left(\sigma\left(\mathfrak{b}^+\right)\right) \hookrightarrow \tilde{\mathbb{X}}(\sigma(\mathfrak{b}))$$

and hence is constant. Indeed, the source is the formal spectrum of a perfect ring and the coordinate ring of the target has no non-trivial maps to perfect rings because its maximal ideal coincides with the nilpotent radical. Therefore, we find that the Lie bracket

$$\tilde{\mathbb{X}}(\sigma(\mathfrak{c})) \times \tilde{\mathbb{X}}(\sigma(\mathfrak{c})) \to \tilde{\mathbb{X}}(\mathfrak{a})$$

lands in the kernel of ρ, which is given by $\tilde{\mathbb{X}}(\sigma(\mathfrak{c}))$. This leads to a contradiction using Claim 7.6.1, and so we are done. $\square$

Remark 7.6.2. Proposition 7.4.3 can be thought of as a generalisation of Proposition 6.2 of [18]. Indeed, this proposition states (when $\mathfrak{a}$ has a biextension structure) that the Lie bracket on

$$\frac{\tilde{\mathbb{X}}(\mathfrak{a})}{\tilde{\mathbb{X}}(\mathfrak{b}_1)}$$

is trivial, or equivalently that the biextension is split. This is what suggested to us that Proposition 7.4.3 might be true.

8. Proof of the main theorem and some variants

8.1. **Preliminaries on Hecke operators.** Let (G, X) be a Shimura datum of Hodge type with reflex field E and let p be a prime such that $G_{\mathbb{Q}_p}$ is quasi-split and split over an unramified extension. Let $U_p \subset G(\mathbb{Q}_p)$ be a hyperspecial subgroup and let $U^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup. Let $\mathbf{Sh}_{G,U}/E$ be the Shimura variety of level $U = U^pU_p$ and for a prime $v|p$ of E let $\mathscr{S}_{G,U}/\mathcal{O}_{E,v}$ be the canonical integral model of $\mathbf{Sh}_{G,U}$ constructed in [51] and in [49] for $p = 2$.

Let $\mathrm{Sh}_{G,U}$ be the base change to $\overline{\mathbb{F}}_p$ of this integral canonical model for some choice of map $\mathcal{O}_{E,v} \to \overline{\mathbb{F}}_p$ and let

$$\mathrm{Sh}_{G,U_p} := \varprojlim_{K^p \subset G(\mathbb{A}_f^p)} \mathrm{Sh}_{G,K^pU_p},$$

which is equipped with an action of $G(\mathbb{A}_f^p)$. Note that the map

$$\pi : \mathrm{Sh}_{G,U_p} \to \mathrm{Sh}_{G,U}$$

is a pro-étale $\underline{U}^p$-torsor.

Let $G^{\mathrm{sc}} \to G^{\mathrm{der}}$ be the simply-connected cover of the derived subgroup of G; we will often identify groups like $G^{\mathrm{sc}}(\mathbb{A}_f^p)$ and $G^{\mathrm{sc}}(\mathbb{Q}_\ell)$ with their images in $G(\mathbb{A}_f^p)$ and $G(\mathbb{Q}_\ell)$. Note that $G^{\mathrm{sc}}(\mathbb{A}_f^p)$ acts on Sh_{G,U_p} via the natural map $G^{\mathrm{sc}}(\mathbb{A}_f^p) \to G(\mathbb{A}_f^p)$.

Let $Z \subseteq \mathrm{Sh}_{G,U}$ be a locally closed subvariety and let $\tilde{Z}$ be the inverse image of Z under π. We say that Z is *stable under the prime-to-p Hecke operators*, or that Z is $G(\mathbb{A}_f^p)$*-stable*, if $\tilde{Z}$ is $G(\mathbb{A}_f^p)$-stable. Similarly we say that Z is *stable under the reduced prime-to-p Hecke operators*, or that Z is $G^{\mathrm{sc}}(\mathbb{A}_f^p)$*-stable*, if $\tilde{Z}$ is $G^{\mathrm{sc}}(\mathbb{A}_f^p)$-stable. For $\ell \neq p$ we call Z $G(\mathbb{Q}_\ell)$-stable if $\tilde{Z}$ is $G(\mathbb{Q}_\ell)$-stable.

The *prime-to-p Hecke orbit* of a point $x \in \mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p)$ is defined to be the image in $\mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p)$ of $G(\mathbb{A}_f^p) \cdot \tilde{x}$, for any choice of lift of $\tilde{x} \to \mathrm{Sh}_{G,U_p}(\overline{\mathbb{F}}_p)$. This does not depend on the choice of $\tilde{x}$ since it can be identified with the image in $\mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p)$ of the $G(\mathbb{A}_f^p)$-orbit of $\pi^{-1}(x)$. We define the *reduced prime-to-p Hecke orbit* of a point $x \in \mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p)$ to be the image in $\mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p)$ of the $G^{\mathrm{sc}}(\mathbb{A}_f^p)$ orbit of $\pi^{-1}(x)$. For $\ell \neq p$ we define the ℓ-adic Hecke orbit or $G(\mathbb{Q}_\ell)$-Hecke orbit of a point x to be the image in $\mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p)$ of the $G(\mathbb{Q}_\ell)$ orbit of $\pi^{-1}(x)$.

Remark 8.1.1. For $g \in G(\mathbb{A}_f^p)$ and $K^p \subset U^p$ there is a finite étale correspondence

$$\begin{tikzcd}
 & \mathrm{Sh}_{G,U_p(K^p \cap gK^pg^{-1})} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
\mathrm{Sh}_{G,U_pK^p} & & \mathrm{Sh}_{G,U_pgK^pg^{-1}} \arrow[r, "g"] & \mathrm{Sh}_{G,U_pK^p} .
\end{tikzcd}$$

and the Hecke operator attached to g is $g \circ p_2 \circ p_1^{-1}$. A locally closed subvariety $Z \subset \mathrm{Sh}_{G,K^p}$ is stable under the Hecke operator attached to g if and only if $\tilde{Z}$ is stable under the action of g considered as an element of $G(\mathbb{A}_f^p)$.

8.2. **Local stabiliser principle.** Choose a Hodge embedding $(G, X) \to (\mathcal{G}_V, \mathcal{H}_V)$ as in Section 5.4. In particular, there is a $\mathbb{Z}_{(p)}$-lattice $V_{(p)}$ on which ψ is $\mathbb{Z}_{(p)}$-valued, such that K_p is the stabiliser in $G(\mathbb{Q}_p)$ of $V_p := V_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$. Then for every sufficiently small compact open subgroup $U^p \subset G(\mathbb{A}_f^p)$, we can find $\mathcal{U}^p \subset \mathcal{G}_V(\mathbb{A}_f^p)$ and a closed immersion

$$\mathrm{Sh}_{G,U} \hookrightarrow \mathcal{A}_{g,U_V}$$

Fix a point $x \in \mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p)$ such that $Y = A_x[p^\infty]$ is completely slope divisible. We write $[\![b]\!] := [\![b_x]\!]$ for the $\mathcal{G}(\breve{\mathbb{Z}}_p)$-$\sigma$-conjugacy class of elements of $G(\breve{\mathbb{Q}}_p)$ associated to x. Let $C_{G,[\![b]\!]} \subset \mathrm{Sh}_{G,U,[b]}$ be associated central leaf with completely slope divisible p-divisible. Then we have seen that the profinite group $\mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p)$ acts on $\mathbf{Def}_{\mathrm{sus}}(Y)$.

For $x \in \mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p)$ we let I_x be the algebraic group over $\mathbb{Q}$ consisting of tensor preserving self-quasi-isogenies of the abelian variety A_x, introduced by Kisin in Section 2.1.2 of [52]. By definition it is a closed subgroup of the algebraic group Aut_x over $\mathbb{Q}$, whose R-points are given by

$$\mathrm{Aut}_x(R) = \left(\mathrm{End}_{\overline{\mathbb{F}}_p}(A_x) \otimes_{\mathbb{Z}} R\right)^\times .$$

We let $I_x(\mathbb{Z}_{(p)}) \subset I_x(\mathbb{Q})$ be the intersection of $I_x(\mathbb{Q})$ with $\left(\mathrm{End}_{\overline{\mathbb{F}}_p}(A_x) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}\right)^\times$. Then for a lift $\tilde{x} \in \mathrm{Sh}_{G,U_p}(\overline{\mathbb{F}}_p)$ of x the stabiliser of $\tilde{x}$ in $G(\mathbb{A}_f^p)$ is equal to $I_x(\mathbb{Z}_{(p)}) \subset I_x(\mathbb{Q}) \subset G(\mathbb{A}_f^p)$, by Lemma 6.1.3 of [39]. The following result is Proposition 6.1.1 of [39], see also Theorem 9.5 of [16] for the Siegel case. Recall that $C_{G,[\![b]\!]}^{/x}$ admits a closed immersion into $\mathbf{Def}_{\mathrm{sus}}(Y)$.

Proposition 8.2.1 (Local stabiliser principle)**.** *If $Z \subset C_{G,[\![b]\!]}$ is a $G(\mathbb{A}_f^p)$-stable reduced closed subset containing x, then $Z^{/x} \subset C_{G,[\![b]\!]}^{/x}$ is stable under the action of $I_x(\mathbb{Z}_{(p)}) \subset \mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p)$ on $\mathbf{Def}_{\mathrm{sus}}(Y)$.*

Remark 8.2.2. The same proof shows that for any $G^{\mathrm{sc}}(\mathbb{A}_f^p)$-stable reduced closed subset $Z \subset C_{G,[\![b]\!]}$ containing x, the subscheme $Z^{/x} \subset C_{G,[\![b]\!]}^{/x}$ is stable under the action of $I_x(\mathbb{Z}_{(p)}) \cap G^{\mathrm{sc}}(\mathbb{A}_f^p)$ on $\mathbf{Def}_{\mathrm{sus}}(Y)$.

8.3. **Proof of Theorem I.** We keep the notation as in Section 8.1 and Section 8.2. Let $G^{\mathrm{ad}} = G_1 \times \cdots \times G_n$ be a decomposition of G^{ad} into a product of $\mathbb{Q}$-simple groups. For a reductive group $\mathbb{G}$ over $\mathbb{Q}_p$ we denote by $B(\mathbb{G})$ the set of σ-conjugacy classes in $\mathbb{G}(\breve{\mathbb{Q}}_p)$.

Definition 8.3.1 (Definition 5.3.2 of [55])**.** An element $[b] \in B(G_{\mathbb{Q}_p})$ is called $\mathbb{Q}$-*non-basic* if the image $[b_i]$ of $[b]$ in $B(G_{i,\mathbb{Q}_p})$ is non-basic for all i.

Let $C_{G,[\![b]\!]} \subseteq \mathrm{Sh}_{G,U,[b]}$ be a central leaf defined as in Section 5.4 and let ν_b be the Newton cocharacter of b for some $b \in [b]$, see [54, Section 1.1.2] for the definition of the Newton cocharacter. Let $P_{\nu_b} \subset G_{\breve{\mathbb{Q}}_p}$ be the associated parabolic subgroup with unipotent radical U_{ν_b}.

Theorem 8.3.2. *Suppose that b is $\mathbb{Q}$-non-basic and let $Z \subset C_{G,[\![b]\!]}$ be a $G(\mathbb{A}_f^p)$-stable closed subvariety. If p is greater than the nilpotency class of the nilpotent Lie algebra* $(\mathrm{Lie}\, U_{\nu_b}, \mathrm{Ad}\, b\sigma)$*, then $Z = C_{G,[\![b]\!]}$.*

The assumption in Theorem I that the Coxeter number satisfies $h(G) \leq p$ implies that p is greater than the nilpotency class of $\operatorname{Lie} U_{\nu_b}$ by Claim 2.1.1 of [45] together with Proposition 31 in Section VI.1.11 of [7]. Therefore Theorem 8.3.2 implies Theorem I.

The discussion in Section 1.6 of [55] implies that the theorem is also true when $[b]$ is $\mathbb{Q}$-basic, that is, when $[b_i]$ is basic for all i. In this case the central leaves are finite and the claim is that $G(\mathbb{A}_f^p)$ acts transitively on them. We expect Theorem 8.3.2 to be true for arbitrary b, but we don't know how to prove the discrete part.

Proof. We reduce immediately to the case that Z is the Zariski closure of the $G(\mathbb{A}_f^p)$-orbit of a point. It follows from Lemma 3.1.2 of [39] that such a Z is itself $G(\mathbb{A}_f^p)$-stable.

By Theorem C of [55], which states that under our assumptions a $G(\mathbb{A}_f^p)$-stable subvariety $Z \subseteq C_{G,[\![b]\!]}$ intersects each connected component of $C_{G,[\![b]\!]}$ non-trivially, it suffices to show that Z is equidimensional of the same dimension as $C_{G,[\![b]\!]}$.

Proposition 2.4.5 of [47] tells us that there is a central leaf $C_{G,[\![b]\!]} \subseteq \mathrm{Sh}_{G,U,[b]}$ such that the universal p-divisible group over $C_{G,[\![b]\!]}$ is completely slope divisible. Since $C_{G,[\![b]\!]}$ and $C_{G,[\![b]\!]}$ share a $G(\mathbb{A}_f^p)$-equivariant finite étale cover, it suffices to this equidimensionality for $C_{G,[\![b']\!]}$ and therefore we will assume without loss of generality that the universal p-divisible group over $C_{G,[\![b]\!]}$ is completely slope divisible. It follows from Lemma 3.1.1 of [39] that the smooth locus Z^{sm} of Z is also $G(\mathbb{A}_f^p)$-stable. It is moreover explained in Section 3.3 of [39] that the abelian variety over $\mathcal{A}_{g,U_V}$ induces an F-isocrystal $\mathcal{M}$ over $\mathrm{Sh}_{G,U}$.

The assumption that b is $\mathbb{Q}$-non-basic allows us to invoke Corollary 3.3.5 of [39][10], which tells us that the unipotent radical of the monodromy group of $\mathcal{M}$ over Z^{sm} is isomorphic the unipotent radical of P_{ν_b}. Theorem II then tells us that for $x \in Z^{\mathrm{sm}}(\overline{\mathbb{F}}_p)$ the monodromy of the isocrystal $\mathcal{M}$ over $\operatorname{Spec} \widehat{\mathcal{O}}_{Z,x}$ is equal to U_{ν_b}.

The assumption that the universal p-divisible group over $C_{G,[\![b]\!]}$ is completely slope divisible tells us that $Z^{/x} \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$ for a completely slope divisible group Y. Theorem 6.1.1 tells us that $Z^{/x}$ is not contained in $Z(\mathfrak{b}^+)$ for any F-stable Lie algebra $\mathfrak{b} \subsetneq \mathfrak{a} = \mathbb{D}(\mathcal{H}_Y^{G,\circ})[\frac{1}{p}] = \operatorname{Lie} U_{\nu_b}$.

Proposition 8.2.1 tells us that $Z^{/x}$ is stable under the action of

$$I_x'(\mathbb{Z}_{(p)}) \subset \mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p).$$

By continuity it is also stable under its closure $\Gamma \subset \mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p)$. It follows as in the proof of Corollary 6.1.6 of [39] that Γ acts strongly non-trivially on $C_{G,[\![b]\!]}^{/x} = Z(\mathfrak{a}^+)$. Therefore Theorem 7.1.1 tells us that $Z^{/x} = Z(\mathfrak{b}^+)$ for some F-stable Lie algebra $\mathfrak{b} \subseteq \mathfrak{a}$, and the previous paragraph tells us that $\mathfrak{a} = \mathfrak{b}$. In other words, $Z^{/x} = C_{G,[\![b]\!]}^{/x}$ for all points $x \in Z^{\mathrm{sm}}(\overline{\mathbb{F}}_p)$. Since $Z^{\mathrm{sm}} \subseteq Z$ is dense because Z is reduced, it follows that Z is equidimensional of the same dimension as $C_{G,[\![b]\!]}$, and therefore we are done. □

8.4. **Isogeny classes are dense in Newton strata.** Let (G, X) be a Shimura variety of Hodge type, let $x \in \mathrm{Sh}_{G,U,[b]}(\overline{\mathbb{F}}_p)$ and let $\mathscr{I}_x \subseteq \mathrm{Sh}_{G,U,[b]}(\overline{\mathbb{F}}_p)$ be the isogeny class of x in the sense of [52].

[10] Hypothesis 2.3.1 of [39] which is assumed to hold in the statement of Corollary 3.3.5 of [39] is true because K_p is hyperspecial, see Lemma 2.3.2 of [39]

Theorem 8.4.1. *Let $W \subset \mathrm{Sh}_{G,U,[b]}$ be a $G(\mathbb{A}_f^p)$ closed subset of a $\mathbb{Q}$-non-basic Newton stratum $\mathrm{Sh}_{G,U,[b]} \subset \mathrm{Sh}_{G,U}$. If p is greater than the nilpotency class of the nilpotent Lie algebra $\mathrm{Lie}\, U_{\nu_b}$, then $W = \mathrm{Sh}_{G,U,[b]}$.*

Proof. The isogeny class $\mathscr{I}_x \subseteq \mathrm{Sh}_{G,U,[b]}(\overline{\mathbb{F}}_p)$ intersects every central leaf $C_{G,[\![b']\!]} \subset \mathrm{Sh}_{G,U,[b]}$ by the Rapoport–Zink uniformisation of isogeny classes (which follows from the main result of [52], see Section 1.4 of op. cit.). Thus if $W \subseteq \mathrm{Sh}_{G,U,[b]}$ is a $G(\mathbb{A}_f^p)$-stable closed subset containing $\mathscr{I}_x \subseteq \mathrm{Sh}_{G,U,[b]}(\overline{\mathbb{F}}_p)$, then W intersects every central leaf $C_{G,[\![b']\!]} \subset \mathrm{Sh}_{G,U,[b]}$ in a $G(\mathbb{A}_f^p)$-stable *non-empty* closed subset W_C. Theorem 8.3.2 now tells us that $W_C = C$ and since $\mathrm{Sh}_{G,U,[b]}$ is the (set-theoretic) union of all the central leaves it follows that $W = \mathrm{Sh}_{G,U,[b]}$. □

8.5. **Orthogonal Shimura varieties.** Conjecture 8.2 of [8] predicts that prime-to-p Hecke orbits are Zariski dense in certain Newton strata of certain *orthogonal* Shimura varieties. These are the Shimura varieties for the group $\mathrm{SO}(M)$ where M is a quadratic space over $\mathbb{Q}$ with signature $(2, m-2)$; they are of abelian type by Appendix B of [60].

In general one does *not* expect that prime-to-p Hecke orbits are Zariski dense in Newton strata, but when the Shimura datum is fully Hodge–Newton decomposable at p, then $\mathbb{Q}$-non-basic Newton strata are equal to central leaves by Theorem E.(2) of [71]. Theorem D of [32] tells us that the orthogonal Shimura varieties in question are indeed fully Hodge–Newton decomposable at p.

It follows from the results of [55] and the proof of Theorem 6.0.7 of [40] that

$$\pi_0(\mathrm{Sh}_{G,U,[b]}) \to \pi_0(\mathrm{Sh}_{G,U})$$

is a bijection for $\mathbb{Q}$-non-basic b for Shimura varieties of Hodge type. Since Newton strata behave well with respect to the dévissage from Hodge type to abelian type, this result also hold for Shimura varieties of abelian type (see Section 5.5 of [72]). Thus Conjecture 8.2 of [8] comes down to showing that the Zariski closure of prime-to-p Hecke orbits have the correct dimensions, and this can be reduced to the Hodge type case and then to Theorem 8.3.2 as in the proof of Corollary 6.4.1 of [39].

The assumption on the Coxeter number comes down to the assumption that $p \geq 2(m-1)$. In the required applications to K3 surfaces it suffices to assume that the number of slopes of the K3-isocrystal is less than or equal to p. Since K3-isocrystals have at most three slopes, this means that it suffices to assume that $p \geq 3$.

Remark 8.5.1. This line of reasoning shows more generally that the Hecke orbit conjecture holds for fully Hodge–Newton decomposable Shimura varieties of abelian type, at primes of hyperspecial good reduction that satisfy the condition $p > n(\mathrm{Lie}\, U_{\nu_b})$, for central leaves in Newton strata corresponding to $\mathbb{Q}$-non-basic $[b]$.

8.6. **ℓ-power Hecke orbits.** In this section we study the Zariski closures of ℓ-adic Hecke orbits of points for primes $\ell \neq p$. Since the ℓ-adic Hecke operators do not, generally, act transitively on $\pi_0(\mathrm{Sh}_{G,U})$, all we can hope to prove is that ℓ-adic Hecke orbits are dense in a union of connected components of a central leaf. However, this cannot be true if $G_{\mathbb{Q}_\ell}$ is totally anisotropic because then there aren't enough ℓ-power Hecke operators. Let $C_{G,[\![b]\!]} \subset \mathrm{Sh}_{G,U,[b]}$ be a central leaf as before and assume that b is $\mathbb{Q}$-non-basic.

Theorem 8.6.1. *Assume that $\mathrm{Sh}_{G,U}$ is proper, that $G_{\mathbb{Q}_\ell}$ is totally isotropic and that p is greater than the nilpotency class of the nilpotent Lie algebra $\mathrm{Lie}\, U_{\nu_b}$. Then any $G(\mathbb{Q}_\ell)$-stable reduced closed subscheme $Z \subset C_{G,[\![b]\!]}$ is a union of connected components of C.*

We start by proving a lemma, cf. Lemma 3.3.2 of [78].

Lemma 8.6.2. *Let $Z \subset \mathrm{Sh}_{G,U}$ be a finite scheme that is $G(\mathbb{Q}_\ell)$-stable. Then Z is contained in the basic locus of $\mathrm{Sh}_{G,U}$.*

Proof. Let $\tilde{x} \in \mathrm{Sh}_{G,U_p}(\overline{\mathbb{F}}_p)$ with image $x \in Z(\overline{\mathbb{F}}_p)$. Let $I_x(\mathbb{Z}_{(p)}) \subset G(\mathbb{A}_f^p)$ be the group of tensor-preserving automorphisms as in Section 8.2. Let U_ℓ be the image of U^p under $G(\mathbb{A}_f^p) \to G(\mathbb{Q}_\ell)$ and identify $I_x(\mathbb{Z}_{(p)})$ with its image under $G(\mathbb{A}_f^p) \to G(\mathbb{Q}_\ell)$. Then the ℓ-adic Hecke orbit of x can be written as

$$I_x(\mathbb{Z}_{(p)})\backslash G(\mathbb{Q}_\ell)/U_\ell,$$

which is finite by assumption. Since the closure of $I_x(\mathbb{Z}_{(p)})$ has finite index in $I_x(\mathbb{Q}_\ell)$, it follows that

$$I_x(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)$$

is compact. Since $I_{x,\mathbb{Q}_\ell}$ is connected it follows from Propositions 8.4 and 9.3 of [6] that it contains a parabolic subgroup of $G_{\mathbb{Q}_\ell}$ and because it is reductive it follows that $I_{x,\mathbb{Q}_\ell} = G_{\mathbb{Q}_\ell}$. It is well known that this only happens when x is contained in the basic locus. □

Lemma 8.6.3. *Assume that $\mathrm{Sh}_{G,U}$ is proper. If a reduced closed subscheme $Z \subseteq \mathrm{Sh}_{G,U,[b]}$ is stable under the action of $G(\mathbb{Q}_\ell)$ for some $\ell \neq p$ such that G^{sc} is totally isotropic, then Z is stable under the action of $G^{\mathrm{sc}}(\mathbb{A}_f^p)$.*

Proof. The proof is almost exactly the same as the proof of Proposition 4.6 of [12]. Nevertheless, we will give a complete proof for the benefit of the reader.

Step 1: A standard argument (see e.g. Proposition 3.3.1 of [78]) using the quasi-affineness of the Ekedahl-Oort stratification (Corollary I.2.6 of [31]) and the properness of $\mathrm{Sh}_{G,U}$ will show that the Zariski closure $\overline{Z}$ of Z in $\mathrm{Sh}_{G,U}$ will contain a point $x \in \mathrm{Sh}_{G,U}(\overline{\mathbb{F}}_p)$ with finite ℓ-power Hecke orbit. Let $Z' \subset Z$ be the union of irreducible components V of Z whose closure $\overline{V}$ does not contain a point with a finite ℓ-power orbit. Suppose for the sake of contradiction that Z' is non-empty, then Z' is a $G(\mathbb{Q}_\ell)$-stable reduced locally closed subscheme of $\mathrm{Sh}_{G,U}$ and thus $\overline{Z'}$ contains a point with a finite ℓ-power orbit, which is a contradiction.

It follows from Lemma 8.6.2 that this implies that x is contained in the basic locus of $\mathrm{Sh}_{G,U}$, and moreover that I_x is an inner form of G (see Corollary 5.2.11 of [35]). Strong approximation away from ∞, ℓ (see Theorem 7.8 of [66]), using the fact that $G_{\mathbb{Q}_\ell}$ is totally isotropic, tells us that the image of

$$I_x(\mathbb{Z}_{(p)}) \to G^{\mathrm{sc}}(\mathbb{A}_f^{p,\ell})$$

is dense. Since the inclusion of the ℓ-adic Hecke orbit of x inside the prime-to-p Hecke orbit of x can be identified with

$$I_x(\mathbb{Z}_{(p)})\backslash G(\mathbb{Q}_\ell)U^p/U^p \subset I_x(\mathbb{Z}_{(p)})\backslash G(\mathbb{A}_f^p)/U^p,$$

we see that the ℓ-adic Hecke orbit of x is $G^{\mathrm{sc}}(\mathbb{A}_f^p)$-stable. Moreover the local stabiliser principle, see Remark 8.2.2, tells us that

$$\overline{Z}^{/x} \subset C_{G,[\![b]\!]}^{/x}$$

will be stable under the action of

$$I_x(\mathbb{Z}_{(p)}) \cap G^{\mathrm{sc}}(\mathbb{A}_f^p) \subset \mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p)$$

and by continuity also stable under its closure.

Step 2: Take a λ-adic Hecke operator $g_\lambda \in G^{\mathrm{sc}}(\mathbb{Q}_\lambda)$ for $\lambda \neq p, \ell$ and let W be the image of $\overline{Z}$ under g_λ. Since every irreducible component of $\overline{Z}$ contains a point x with finite ℓ-power Hecke orbit, it follows that every irreducible component of W contains an element in the g_λ-orbit of such an x. Since $\overline{Z}$ contains the $G^{\mathrm{sc}}(\mathbb{A}_f^p)$ orbit of x, it follows that every irreducible component W_i of W intersects $\overline{Z}$ in a point y_i with finite ℓ-power Hecke orbit.

The reduced prime-to-p Hecke orbit of y_i has the form

$$I_{y_i}(\mathbb{Z}_{(p)})^{\mathrm{sc}} \backslash G^{\mathrm{sc}}(\mathbb{A}_f^p) U^p / U^p.$$

By strong approximation (Theorem 7.8 of [66]) away from ℓ for $I_{y_i}^{\mathrm{sc}}$, using the fact that $G_{\mathbb{Q}_\ell}$ is totally isotropic, we can choose $\delta \in I_{y_i}^{\mathrm{sc}}(\mathbb{Q})$ which lands in $U^{p,\ell,\lambda}$ and such that there is an element $g_\ell \in G^{\mathrm{sc}}(\mathbb{Q}_\ell)$ such that $\delta \cdot g_\ell = g_\lambda^{-1}$ in g_λ in $I_{y_i}(\mathbb{Z}_{(p)}) \backslash G^{\mathrm{sc}}(\mathbb{A}_f^p) U^p / U^p$.

We see that $g_\lambda \circ g_\ell$ fixes y_i, and since $\overline{Z}$ is g_ℓ-stable, the image of $\overline{Z}$ under $g_\lambda \circ g_\ell$ is equal to W. Now we consider the closed subschemes

$$W_i^{/y_i}, \overline{Z}^{/y_i} \subseteq \mathrm{Sh}_{G,U}^{/y_i}.$$

The subscheme $W_i^{/y_i}$ is the image of $\overline{Z}^{/y_i}$ under the Hecke operator $g_\lambda \circ g_\ell$. Since $\overline{Z}^{/y_i}$ is stable under the action of the closure of $I_x(\mathbb{Z}_{(p)})$ and since $g_\lambda \circ g_\ell$ is contained in that closure by construction, it follows that

$$W_i^{/y_i} \subset \overline{Z}^{/y_i}.$$

From this we deduce that $W_i \subset \overline{Z}$. Thus every irreducible component W_i of W is contained in $\overline{Z}$, and we conclude that $\overline{Z}$ is stable under the action of g_λ. Since λ and g_λ were arbitrary, it follows that $\overline{Z}$ is stable under the action of $G^{\mathrm{sc}}(\mathbb{A}_f^p)$.

We know that Z is the intersection of $\overline{Z}$ with $\mathrm{Sh}_{G,U,[b]}$ and as the intersection of two $G^{\mathrm{sc}}(\mathbb{A}_f^p)$-stable subschemes is must itself be $G^{\mathrm{sc}}(\mathbb{A}_f^p)$-stable. $\square$

Proof of Theorem 8.6.1. If we let $\mathcal{M}$ be the isocrystal attached to the universal abelian variety over Z and if we let $x \in Z$ be a smooth point, then as in the proof of Theorem 8.3.2 we can combine Corollary 3.3.5 of [39] with Theorem II to deduce that the monodromy of the isocrystal $\mathcal{M}$ over $\operatorname{Spec} \widehat{\mathcal{O}}_{Z,x}$ is isomorphic to U_{ν_b}.

Proposition 8.2.1, see Remark 8.2.2, tells us that $Z^{/x}$ is stable under the action of

$$\left(I_x(\mathbb{Z}_{(p)}) \cap G^{\mathrm{sc}}(\mathbb{A}_f^p)\right) \subset \mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p).$$

By continuity it is also stable under the closure $\Gamma \subset \mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p)$. It follows as in the proof of Corollary 6.1.6 of [39] that Γ acts strongly non-trivially on $C_{G,[\![b]\!]}^{/x} = Z(\mathfrak{a}^+)$. The same argument as in the proof of Theorem 8.3.2 allows us to conclude that Z is a union of connected components of C. $\square$

8.7. **Further questions of Chai–Oort.** Let $Z \subseteq C_{(Y,\lambda)}$ be an irreducible smooth closed subvariety and let $x \in Z(\overline{\mathbb{F}}_p)$. We call Z *strongly Tate-linear at* x if $Z^{/z} \subseteq C_{(Y,\lambda)}^{/x}$ is a strongly Tate-linear subvariety.

Question 8.7.1. Suppose that Z is strongly Tate-linear at some closed point $x_0 \in Z(\overline{\mathbb{F}}_p)$. Is Z then strongly Tate-linear at all closed point $x \in Z(\overline{\mathbb{F}}_p)$?

It follows from Theorem II that the monodromy group of $\mathcal{M}$ over $\operatorname{Spec} \widehat{\mathcal{O}}_{Z,x}$ does not depend on x. Now the validity of Conjecture 5.6.2 would imply that the question above has an affirmative answer.

Question 8.7.2. Suppose that Z is strongly Tate-linear at some closed point $x_0 \in Z(\overline{\mathbb{F}}_p)$. Must Z then be an irreducible component of a central leaf in the mod p reduction of a Shimura variety of Hodge type?

The stronger assertion that Z must itself be an irreducible component of a Shimura variety of Hodge type is false in general, because only finitely many central leaves in a given Newton stratum contain the mod p reductions of special points by Theorem 1.3 of [53]. We note that if $C_{(Y,\lambda)}$ is the ordinary locus and Z is proper, then this stronger assertion *is* true by work of Moonen [61].

8.8. **Results at ramified primes and parahoric level.** The statements of out theorems make sense for central leaves in special fibres of the Kisin–Pappas [50] integral models of Shimura varieties of parahoric level at tamely ramified primes.

8.8.1. If the group is unramified and the level is parahoric, then the Hecke orbit conjecture for central leaves at parahoric level follows immediately from Theorem 8.3.2. The main observation is that the forgetful map

$$C' \to C,$$

where C' is a central leaf at Iwahori level and C is a central leaf at hyperspecial level, is equivariant for the prime-to-p Hecke operators and induces a bijection on π_0. This last statement can be proven using the surjectivity of $C' \to C$[11], and the explicit description of connected components of Igusa varieties in [40, 55].

Rapoport–Zink uniformisation of isogeny classes at parahoric level[11], implies as before that isogeny classes are dense in the Newton strata containing them.

8.8.2. If the group is ramified, then it is not always true that $G(\mathbb{A}_f^p)$ acts transitively on $\pi_0(\mathrm{Sh}_{G,U})$ (see [63] for explicit counterexamples) and therefore it is not necessarily true that $G(\mathbb{A}_f^p)$ acts transitively on $\pi_0(C)$ either because $\pi_0(C) \to \pi_0(\mathrm{Sh}_{G,U})$ is surjective. Nevertheless, we expect that the continuous part of the Hecke orbit conjecture is true for ramified groups. In fact, we suspect that the strategy adopted in this paper can be made to work for ramified groups.

Appendix A. Complete restricted perfections, after Chai–Oort

In this section we will collect some results about completed perfections and complete restricted perfections of complete Noetherian local rings over a perfect field κ of characteristic p. For the latter part we will follow Chai–Oort [18].

A.1. **Completed perfections.** Throughout this section we let will consider complete Noetherian local κ-algebras with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = \kappa$, the main example being the ring of power series $R = \kappa[[X_1, \cdots, X_n]]$. We let $\varphi_{\mathrm{rel}} : R \to R$ denote the relative Frobenius homomorphism over κ.

Definition A.1.1. The (relative) perfection R^{perf} of R over κ is the (filtered) colimit

$$\varinjlim_{i \in \mathbb{Z}} R_i,$$

where $R_i = R$ and the transition maps $R_i \to R_{i+1}$ are given by φ_{rel}.

[11] This surjectivity is axiom 4c of the He-Rapoport axioms, see [71], and follows from Rapoport–Zink uniformisation at parahoric level, which is Theorem 2 of [41]

Example A.1.2. When $R = \kappa[[X_1, \cdots, X_n]]$, we get the ring

$$\bigcup_m \kappa[[X_1^{1/p^m}, \cdots, X_n^{1/p^m}]].$$

When $n = 1$ we note that this ring does not contain the element

$$\sum_{i=1}^{\infty} X^i \cdot X^{1/p^i},$$

while the partial sums do converge in the $\mathfrak{m}$-adic topology. In other words, R^{perf} is not $\mathfrak{m}$-adically complete.

Definition A.1.3. Let $(R, \mathfrak{m})$ be a complete Noetherian local κ-algebra with residue field κ. Then we define $R^{\mathrm{perf},\wedge}$ to be the $I = \mathfrak{m}.R^{\mathrm{perf}}$-adic completion of R^{perf}.

Lemma A.1.4. *The ring $R^{\mathrm{perf},\wedge}$ is an I-adically complete local κ-algebra with residue field κ.*

Proof. Lemma 05GG of [73] tells us that it is complete, because I is finitely generated. To show that it is a local ring it suffices to show that $R^{\mathrm{perf}}/I = R^{\mathrm{perf},\wedge}/IR^{\mathrm{perf},\wedge}$ is a local ring because every maximal ideal of $R^{\mathrm{perf},\wedge}$ contains $IR^{\mathrm{perf},\wedge}$ by Lemma 05GI of [73], and R^{perf}/I is a local ring because R^{perf} is. □

Lemma A.1.5. *Let $(R, \mathfrak{m})$ be a complete Noetherian local κ-algebra with residue field κ. Then there is a natural isomorphism of topological rings*

$$\varprojlim_{\varphi_{\mathrm{rel}}} \left(R^{\mathrm{perf}}/\mathfrak{m}R^{\mathrm{perf}}\right) \simeq R^{\mathrm{perf},\wedge}.$$

In particular $R^{\mathrm{perf},\wedge}$ is perfect.

Proof. There is an isomorphism of inverse systems

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\varphi_{\mathrm{rel}}} & R^{\mathrm{perf}}/\mathfrak{m}R^{\mathrm{perf}} \cdots & \xrightarrow{\varphi_{\mathrm{rel}}} & R^{\mathrm{perf}}/\mathfrak{m}R^{\mathrm{perf}} & \xrightarrow{\varphi_{\mathrm{rel}}} & R^{\mathrm{perf}}/\mathfrak{m}R^{\mathrm{perf}} \\
& {\scriptstyle g_2}\big\uparrow & & {\scriptstyle g_1}\big\uparrow & & {\scriptstyle g_0}\big\uparrow \\
\cdots \longrightarrow & R^{\mathrm{perf}}/\varphi_{\mathrm{rel}}^2(\mathfrak{m})R^{\mathrm{perf}} \cdots & \longrightarrow & R^{\mathrm{perf}}/\varphi_{\mathrm{rel}}(\mathfrak{m})R^{\mathrm{perf}} & \longrightarrow & R^{\mathrm{perf}}/\mathfrak{m}R^{\mathrm{perf}},
\end{array}$$

where the transition maps on the bottom are the natural reduction maps. The maps g_i are constructed by precomposing the natural surjection $R^{\mathrm{perf}} \to R^{\mathrm{perf}}/\mathfrak{m}R^{\mathrm{perf}}$ with $\varphi_{\mathrm{rel}}^{-i}$ and noticing that this map is surjective with kernel $\varphi_{\mathrm{rel}}^i(\mathfrak{m})$, giving the isomorphisms g_i.

Because the sequence of ideals $\varphi_{\mathrm{rel}}^i(\mathfrak{m})R^{\mathrm{perf}}$ and $\mathfrak{m}^i R^{\mathrm{perf}} = (\mathfrak{m}R^{\mathrm{perf}})^i$ are cofinal, the inverse limit over the bottom inverse system is canonically isomorphic to $R^{\mathrm{perf},\wedge}$. □

Lemma A.1.6 (cf. Corollary 4.2.3 of [18])**.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local κ-algebra with residue field κ. Then the natural map $R^{\mathrm{perf}} \to R^{\mathrm{perf},\wedge}$ is injective.*

Proof. The lemma comes down to showing that the $\mathfrak{m}$-adic topology on R^{perf} is separated, in other words that

$$\bigcap_k (\mathfrak{m}^k R^{\mathrm{perf}}) = (0).$$

For this we use the following result of Chai–Oort:

Claim A.1.7 (Proposition 4.2.2 of [18])**.** *Let R be a Noetherian local domain whose normalisation in its fraction field is finite over R. Then there exists a positive integer n_0 such that for all integers $a \geq 0$ and for all $n \geq a \cdot n_0$*

$$\{x \in R \,:\, x^a \in \mathfrak{m}^n\} \subseteq \mathfrak{m}^{\lfloor \frac{n}{a} - n_0 \rfloor}.$$

Now y be an element in $\bigcap_k (\mathfrak{m}^k R^{\mathrm{perf}})$, and choose m such that $x = y^{p^m} \in R$. Then for all k we can write

$$y = \sum_{|I|=k} x_I r_I^{1/p^{n(k)+m}},$$

with $r_I \in R$. Here $x_1, \cdots, x_j$ are generators of $\mathfrak{m}$ and for a multiset I of integers between 1 and j we write $x_I = \prod_{i \in I} x_i$. Then clearly

$$x^{p^{n(k)}} = y^{p^{n(k)+m}} = \sum_{|I|=k} (x_I)^{p^{n(k)+m}} r_I \in \mathfrak{m}^{k \cdot (p^{n(k)+m})} R.$$

By the claim for k large enough (larger than n_0, say) we have

$$x \in \mathfrak{m}^{\lfloor k \cdot p^m - k \rfloor}.$$

Since this holds for all sufficiently large k, Krull's intersection theorem tells us that $x = 0$ and therefore $y = 0$. $\square$

A.1.8. If $\alpha : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local homomorphism of complete local Noetherian κ-algebras, then there is an induced homomorphism

$$\alpha^{\mathrm{perf}} : R^{\mathrm{perf}} \to S^{\mathrm{perf}}$$

by functoriality of the perfection construction and then similarly a morphism

$$\alpha^{\mathrm{perf},\wedge} : R^{\mathrm{perf},\wedge} \to S^{\mathrm{perf},\wedge}.$$

Lemma A.1.9 (cf. Corollary 4.2.4 of [18])**.** *Suppose that S is a domain. Then if α is injective then so are α^{perf} and $\alpha^{\mathrm{perf},\wedge}$. In addition, $R^{\mathrm{perf}} \subseteq S^{\mathrm{perf}}$ has the subspace topology.*

Proof. Lemma A.1.6 tells us that we get a commutative diagram with injective horizontal maps

$$\begin{tikzcd}
R \arrow[r, hook] \arrow[d, hook, "\alpha"] & R^{\mathrm{perf}} \arrow[r, hook] \arrow[d, hook, "\alpha^{\mathrm{perf}}"] & R^{\mathrm{perf},\wedge} \arrow[d, "\alpha^{\mathrm{perf},\wedge}"] \\
S \arrow[r, hook] & S^{\mathrm{perf}} \arrow[r, hook] & S^{\mathrm{perf},\wedge}.
\end{tikzcd}$$

The map α^{perf} is injective because S^{perf} is reduced and since elements in R^{perf} are $\overline{\mathbb{F}}_p$-linear combinations of p^n-th roots of elements of R.

Because $R \to S$ is a local homomorphism we find that $\mathfrak{m}^k \subseteq \mathfrak{n}^k \cap R$ and Chevalley's theorem tells us that for each $k \geq 0$ there exists an $n(k)$ such that $\mathfrak{n}^{n(k)} \cap R \subseteq \mathfrak{m}^k$. Therefore the $\mathfrak{m}$-adic topology on R is equal to the subspace topology coming from S. If we can show that R^{perf} has the subspace topology coming from S^{perf} then Lemma 0ARZ of [73] tells us that the induced map on completions is injective.

In other words, we want to show that for each $k \gg 0$ there exists an $n'(k)$ such that

$$(\mathfrak{n}^{n'(k)} S^{\mathrm{perf}}) \cap R^{\mathrm{perf}} \subseteq \mathfrak{m}^k R^{\mathrm{perf}}.$$

Now fix k and let $y \in (\mathfrak{n}^n S^{\mathrm{perf}}) \cap R^{\mathrm{perf}}$, then we are going to choose $n = n'(k)$ such that the above inclusion holds. If we choose m such that $x = y^{p^m} \in R$, then we can write

$$y = \sum_{|I|=n} x_I r_I^{1/p^{t+m}},$$

for some t and some $r_I \in S$. Here $x_1, \cdots, x_j$ is a basis of $\mathfrak{n}$ and for a multiset I of integers between 1 and j we write $x_I = \prod_{i \in I} x_i$. Then clearly

$$x^{p^t} = y^{p^{t+m}} = \sum_{|I|=n} x_I^{p^{t+m}} r_I \in \mathfrak{n}^{n \cdot (p^{t+m})} S.$$

And so x^{p^t} is contained in

$$\mathfrak{n}^{n \cdot (p^{t+m})} \cap R.$$

If we can prove the following claim then the lemma follows:

Claim A.1.10. *For $k \gg 0$ there is a choice of $n = n'(k)$, independent of t and m, such that*

$$\mathfrak{n}^{n \cdot (p^{t+m})} \cap R \subseteq \mathfrak{m}^{(k+1) \cdot p^{t+m}}.$$

Indeed Claim A.1.7 says that if $k \gg 0$ we have

$$x \in \mathfrak{n}^{\lfloor (k+1) \cdot p^m - n_0 \rfloor}.$$

and so if we choose m such that $p^m \gg n_0$ then we find that y is in

$$\mathfrak{m}^k R^{\mathrm{perf}}.$$

$\square$

Proof of Claim A.1.10. We can certainly choose n such that

$$\mathfrak{n}^n \cap R \subseteq \mathfrak{m}^{k+1}$$

and then

$$\mathfrak{n}^{n \cdot (p^{t+m})} \subseteq \varphi_{\mathrm{rel}}^{t+m-c}(\mathfrak{n}^n)$$

where c_0 is a constant only depending on the characteristic p and the number of generators of $\mathfrak{n}$. Therefore

$$\begin{aligned}(\mathfrak{n}^{n \cdot (p^{t+m})} \cap R) &\subseteq \varphi_{\mathrm{rel}}^{t+m-c_0}(\mathfrak{n}^n) \cap R \\ &\subseteq \varphi^{t+m-c_1}(\mathfrak{m}^{k+1}) \\ &\subseteq \mathfrak{m}^{(k+1) \cdot p^{t+m} - c_2}\end{aligned}$$

which is contained in $\mathfrak{m}^{k \cdot p^{t+m}}$ for $k \gg 0$. The constant c_1 only depends on c_0 and $R \to S$ (in particular on the biggests j for which there exists an $x \in \mathfrak{m}$ with $x^{1/p^j} \in S$). $\square$

A.1.11. Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be complete Noetherian local κ-algebras with residue field κ, then their completed tensor product

$$T = R \widehat{\otimes}_\kappa S$$

is again a complete Noetherian local κ-algebra with residue field k.

Lemma A.1.12. *There is a natural continuous isomorphism*

$$T^{\mathrm{perf},\wedge} \to R^{\mathrm{perf},\wedge} \widehat{\otimes}_k S^{\mathrm{perf},\wedge}.$$

Proof. It suffices to show that there is a natural isomorphism of functors

$$\operatorname{Spf} T^{\mathrm{perf},\wedge} = \operatorname{Spf} R^{\mathrm{perf},\wedge} \times \operatorname{Spf} S^{\mathrm{perf},\wedge},$$

which is a straightforward consequence of Lemma A.1.17 below. Indeed, limits commute with products in categories of sheaves and so the perfection of the product of two formal schemes is naturally isomorphic to the product of the perfections. □

A.1.13. If $R = \kappa[[X_1, \cdots, X_n]]$ then

$$R^{\mathrm{perf}} = \bigcup_m \kappa[[X_1^{1/p^m}, \cdots, X_n^{1/p^m}]].$$

and $R^{\mathrm{perf},\wedge}$ is its completion, which we will now describe explicitly. Let S be the ring of formal expressions

$$\sum_{I \in (\mathbb{Z}_{\geq 0}[1/p])^n} a_I x_I$$

with $a_I \in \kappa$ and such that for each $N \in \mathbb{Z}_{\geq 0}$ the set

$$\{I \in (\mathbb{Z}_{\geq 0}[1/p])^n : a_I \neq 0\} \cap [0, N]^n$$

is finite (thus for example the element $x_1^{1/p} + x_1^{1/p^2} + x^{1/p^3} + \cdots$ is not contained in S). The ring structure on S is given by the usual ring structure on power series, which makes sense because of the finiteness condition. It is easy to see that S is an integral domain for example by looking at leading coefficients. It is also clear that S is a perfect κ-algebra, because we are allowed to divide by p in the exponents and this does not change the finiteness condition.

There is a natural injective ring homomorphism $R \to S$ sending X_i to x_i and since S is perfect this extends to an injective ring homomorphism $R^{\mathrm{perf}} \to S$.

Lemma A.1.14. *The ring S is I-adically complete, where $I = (x_1, \cdots, x_n)$.*

Proof. If we are given a sequence $f_1, f_2, \cdots$ of elements in S such that f_{k+m} is congruent to f_k mod I^k for all k, m, then the coefficients of the f_i form a stabilising sequence which leads to an element $f \in S$ which is congruent to f_k mod I^k for all k. To be precise the element f_k determines the coefficients $a_{f,i}$ for $i \in [0, k]^n$ and so we get a well defined limit f. □

The natural map $R^{\mathrm{perf}} \to S$ clearly induces isomorphisms

$$S/I^k S \simeq R^{\mathrm{perf}}/I^k R^{\mathrm{perf}}$$

for all k and therefore $R^{\mathrm{perf}} \to S$ can be identified with the natural map $R^{\mathrm{perf}} \to R^{\mathrm{perf},\wedge}$.

Corollary A.1.15. *When $R = \kappa[[X_1, \cdots, X_n]]$, the completed perfection $R^{\mathrm{perf},\wedge}$ is an integral domain.*

We do not know whether $R^{\mathrm{perf},\wedge}$ is always an integral domain when R is.

A.1.16. Recall our conventions on formal schemes from Section 2.1: Let R be a κ-algebra that is complete with respect to a finitely generated ideal $I \subseteq R$. We write Spf R for functor

$$\varinjlim_{j} \operatorname{Spec} R/I^j R$$

on the category of κ-algebras, where the transition maps are the ones induced by the natural quotient maps $R/I^{j+1} \to R/I^j$.

Lemma A.1.17. *Let $(R, \mathfrak{m})$ be a complete Noetherian local κ-algebra with residue field κ, and consider the inverse system*

$$\cdots \longrightarrow \operatorname{Spf} R \xrightarrow{\varphi_{\mathrm{rel}}} \operatorname{Spf} R \longrightarrow \cdots .$$

Then the inverse limit of this diagram (in the category of functors) is isomorphic to Spf $R^{\mathrm{perf},\wedge}$.

Proof. Let us write Spf $R = \varinjlim \operatorname{Spec} R/\varphi_{\mathrm{rel}}^n(\mathfrak{m})$, then the following diagram is Cartesian

$$\begin{tikzcd}
\operatorname{Spec} R/\varphi_{\mathrm{rel}}^{n+1}(\mathfrak{m}) \arrow[r, hook] \arrow[d] & \operatorname{Spf} R \arrow[d, "\varphi"] \\
\operatorname{Spec} R/\varphi_{\mathrm{rel}}^{n}(\mathfrak{m}) \arrow[r, hook] & \operatorname{Spf} R,
\end{tikzcd}$$

and the left vertical map corresponds to the map

$$R/\varphi_{\mathrm{rel}}^n(\mathfrak{m}) \to R/\varphi_{\mathrm{rel}}^{n+1}(\mathfrak{m})$$

induced by φ, which is injective if R is reduced. The diagram

$$\varinjlim_{m} R/\varphi_{\mathrm{rel}}^{n+m}(\mathfrak{m})$$

where the transition maps are the maps induced by φ can be identified with the diagram

$$\varinjlim_{m} R_m/\varphi_{\mathrm{rel}}^{n}(\mathfrak{m}) R_m$$

where R is the ring R equipped with an R-algebra structure via the map $\varphi_{\mathrm{rel}}^m : R \to R$. It is clear that the limit of this diagram of R-algebras is given by $R^{\mathrm{perf}}/\varphi_{\mathrm{rel}}^n(\mathfrak{m}) R^{\mathrm{perf}}$.

So if $\mathcal{F}$ is the limit of the diagram in the statement of the Lemma, then there is a Cartesian diagram

$$\begin{tikzcd}
\operatorname{Spec} R^{\mathrm{perf}}/\varphi_{\mathrm{rel}}^n(\mathfrak{m} R^{\mathrm{perf}}) \arrow[r, hook] \arrow[d] & \mathcal{F} \arrow[d] \\
\operatorname{Spec} R/\varphi^n(\mathfrak{m}) \arrow[r] & \operatorname{Spf} R.
\end{tikzcd}$$

It follows from this (cf. Lemma 0AJH of [73]) that

$$\mathcal{F} \simeq \varinjlim_{n} \operatorname{Spec} R^{\mathrm{perf}}/\varphi_{\mathrm{rel}}^n(\mathfrak{m} R^{\mathrm{perf}})$$

which proves the lemma. $\square$

A.2. **Complete restricted perfections.** In this section we follow Section 3.5 of [18]. Let $(R, \mathfrak{m})$ be a complete Noetherian local κ-algebra with residue field κ. Let r, s, n_0 be natural numbers with $0 < r < s$ and $n_0 \geq 0$ and consider the following subset of $R^{\mathrm{perf},\wedge}$:

$$R^{\mathrm{perf},\wedge}_{s,r,\geq n_0} := \left\{ x \in R^{\mathrm{perf},\wedge} \mid \text{For } n \geq n_0 \text{ there is an } y \in R \text{ such that } x^{p^{nr}} \equiv y \mod \varphi^{ns}(\mathfrak{m}) R^{\mathrm{perf},\wedge}. \right\}$$

This is clearly a subring because the image of

$$R/\varphi^{ns}_{\mathrm{rel}}(\mathfrak{m}) \to R^{\mathrm{perf},\wedge}/\varphi^{ns}_{\mathrm{rel}}(\mathfrak{m}) R^{\mathrm{perf},\wedge}$$

is a subring. It is moreover closed in the $\mathfrak{m}$-adic topology on $R^{\mathrm{perf},\wedge}$ because it is defined by congruence conditions. Furthermore if $R \to S$ is a local homomorphism then there is clearly an induced ring homomorphism

$$R^{\mathrm{perf},\wedge}_{s,r,\geq n_0} \to S^{\mathrm{perf}}_{s,r,\geq n_0},$$

which is injective if $R \to S$ is injective and S is a domain, by Lemma A.1.9. These rings are called complete restricted perfections because they are complete and sit between R and $R^{\mathrm{perf},\wedge}$, hence the *restricted* perfection.

A.2.1. If $R = \kappa[[X_1, \cdots, X_n]]$ then

$$R^{\mathrm{perf},\wedge}_{s,r,\geq n_0} \subseteq R^{\mathrm{perf},\wedge}$$

can be described as a subring of S as follows. It consists of those formal series

$$\sum_{I \in (\mathbb{Z}_{\geq 0}[1/p])^n} a_I x_I$$

such that for all $n \geq n_0$ the truncated series

$$\sum_{\substack{I \in (\mathbb{Z}_{\geq 0}[1/p])^n \\ |p^{nr} I|_\infty \leq p^{ns}}} a_I x_{p^{nr} I}$$

is an actual power series. Here the norm $| - |_\infty$ on $\mathbb{R}^n$ is the maximum of the norms of the coordinates. An elementary argument shows that this condition is equivalent to the condition that

$$-\operatorname{ord}_p(I) \leq \max \left\{ n_0, s \cdot \left\lfloor \frac{\log_p |I|_\infty}{s - r} \right\rfloor + 1 \right\},$$

where $\operatorname{ord}_p(I)$ is the minimum of the p-adic valuations of the coordinates.

A.2.2. Chai and Oort also define a variant

$$\kappa[[X_1, \cdots, X_n]]^{\mathrm{perf},\wedge}_{E,C;d} \subseteq \kappa[[X_1, \cdots, X_n]]^{\mathrm{perf},\wedge}$$

for real numbers $C > 0, d \geq 0, E > 0$ consisting of those series

$$\sum_{I \in (\mathbb{Z}_{\geq 0}[1/p])^n} a_I x_I$$

satisfying

$$p^{-\operatorname{ord}_p(I)} \leq \max(C \cdot (|I|_\infty + D)^E, 1\}.$$

Remark A.2.3. Chai and Oort use the notation

$$\kappa\langle\langle X_1^{p^{-\infty}}, \cdots, X_n^{p^{-\infty}}\rangle\rangle_{C;d}^{E,\sharp}$$

for these rings; the superscript $\sharp$ is used to distinguish it from a $\flat$-variant

$$\kappa\langle\langle X_1^{p^{-\infty}}, \cdots, X_n^{p^{-\infty}}\rangle\rangle_{C;d}^{E,\flat} \supseteq \kappa\langle\langle X_1^{p^{-\infty}}, \cdots, X_n^{p^{-\infty}}\rangle\rangle_{C;d}^{E,\sharp}.$$

We have opted not to use this notation because it could cause confusion with the $\sharp$ and $\flat$ notation introduced by Scholze to denote tilts and untilts of (semi-)perfect rings.

Chai and Oort prove (Lemma 4.12 of [18]) that for each r, s, n_0 there are choices of C, d, E such that

$$\kappa[[X_1, \cdots, X_n]]_{s,r,\geq n_0}^{\mathrm{perf}} \subseteq \kappa[[X_1, \cdots, X_n]]_{E,C;d}^{\mathrm{perf},\wedge}.$$

These rings are then used to state Proposition of 6.4 of [18], where we note that composition of perfected power series is well defined.

Theorem A.2.4 (Proposition 6.4 of [18], Proposition 10.5.3 of [20])**.** *Let $f(\underline{u}, \underline{v}) \in \kappa[[u_1, \cdots, u_a, v_1, \cdots, v_b]]$ be a formal power series. Let $(g_1(\underline{x}), \cdots, g_a(\underline{x}))$ be an a-tuple of elements in*

$$\kappa[[X_1, \cdots, X_n]]_{E,C;d}^{\mathrm{perf},\wedge},$$

whose degree zero terms vanish, and similarly let $(h_1(\underline{y}), \cdots, h_b(\underline{y}))$ be an b-tuple of elements in

$$\kappa[[Y_1, \cdots, Y_n]]_{E,C;d}^{\mathrm{perf},\wedge},$$

whose degree zero terms vanish. Suppose that there is a sequence of nonnegative integers d_n for $n \in \mathbb{Z}_{\geq 0}$ such that

$$\lim_{n \to \infty} \frac{q^n}{d_n} = 0$$

where $q = p^r$ is some power of p and such that for all n the element

$$f(g_1(\underline{X}), \cdots, g_a(\underline{X}), h_1(\underline{X})^{q^n}, \cdots, h_b(\underline{X})^{q^n})$$

is congruent to zero modulo $(X_1, \cdots, X_n)^{d_n}$. Then

$$f(g_1(\underline{X}), \cdots, g_a(\underline{X}), h_1(\underline{Y}), \cdots, h_b(\underline{Y})) = 0.$$

Proof. This is a special case of Proposition 6.4 of [18] (cf. the remark after its statement); strictly speaking this proposition is stated for the $\flat$-versions of the complete restricted perfections of Chai–Oort, but those contain the $\sharp$-versions (see A.2.3). The result is also stated as Proposition 10.5.3 in [20]. $\square$

Remark A.2.5. Let us recall from Section 6.1 of [18] some intuition behind Theorem A.2.4 in the case that the g_i and h_i are ordinary power series. Define

$$U = \operatorname{Spf} \kappa[[u_1, \cdots, u_a]],\ V = \operatorname{Spf} \kappa[[v_1, \cdots, v_b]],\ W = \operatorname{Spf} \kappa[[x_1, \cdots, x_n]].$$

Then $f(u, v)$ corresponds to a global function f on $U \times V$, and $(g_1, \cdots, g_a), (h_1, \cdots, h_b)$ correspond to morphisms

$$g : W \to U$$
$$h : W \to V.$$

The theorem can then be interpreted as saying that $(g,h)^*(f)$ is trivial if and only if for all n the global function

$$(1, \varphi_{\mathrm{rel}}^n)^* \circ (g,h)^*(f)$$

is trivial mod $(X_1, \cdots, X_n)^{d^n}$ for a sequence of integers d_n satisfying

$$\lim_{n\to\infty} \frac{q^n}{d_n} = 0.$$

References

[1] Johannes Anschütz and Arthur-César Le Bras, *Prismatic Dieudonné theory*, arXiv e-prints (July 2019), available at `1907.10525`.

[2] Pierre Berthelot and William Messing, *Théorie de Dieudonné cristalline. I*, Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. I, 1979, pp. 17–37. MR563458

[3] Bhargav Bhatt and Jacob Lurie, *The prismatization of p-adic formal schemes*, arXiv e-prints (January 2022), available at `2201.06124`.

[4] Bhargav Bhatt, Matthew Morrow, and Peter Scholze, *Topological Hochschild homology and integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **129** (2019), 199–310. MR3949030

[5] Bhargav Bhatt and Peter Scholze, *Prismatic F-crystals and crystalline Galois representations*, arXiv e-prints (June 2021), available at `2106.14735`.

[6] Armand Borel and Jacques Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–150. MR207712

[7] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley. MR1890629

[8] Daniel Bragg and Ziquan Yang, *Twisted Derived Equivalences and Isogenies between K3 Surfaces in Positive Characteristic*, arXiv e-prints (February 2021), available at `2102.01193`.

[9] Ana Caraiani and Peter Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties*, Ann. of Math. (2) **186** (2017), no. 3, 649–766. MR3702677

[10] Ching-Li Chai, *Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli*, Invent. Math. **121** (1995), no. 3, 439–479. MR1353306

[11] ———, *Families of ordinary abelian varieties: canonical coordinates, p-adic monodromy, tate-linear subvarieties and hecke orbits*, 2003. preprint available at `https://www2.math.upenn.edu/~chai/papers_pdf/fam_ord_av.pdf`.

[12] ———, *Monodromy of Hecke-invariant subvarieties*, Pure Appl. Math. Q. **1** (2005), no. 2, Special Issue: In memory of Armand Borel. Part 1, 291–303. MR2194726

[13] ———, *Hecke orbits as Shimura varieties in positive characteristic*, International Congress of Mathematicians. Vol. II, 2006, pp. 295–312. MR2275599

[14] ———, *A rigidity result for p-divisible formal groups*, Asian J. Math. **12** (2008), no. 2, 193–202. MR2439259

[15] ———, *Sustained p-divisible groups and a foliation on moduli spaces of abelian varieties*, Proceedings of the International Consortium of Chinese Mathematicians 2018, [2020] ©2020, pp. 3–30. MR4251135

[16] Ching-Li Chai and Frans Oort, *Moduli of abelian varieties and p-divisible groups*, Arithmetic geometry, 2009, pp. 441–536. MR2498069

[17] ———, *Monodromy and irreducibility of leaves*, Ann. of Math. (2) **173** (2011), no. 3, 1359–1396. MR2800716

[18] ———, *Rigidity for biextensions of formal groups* (2017). Available at `https://www2.math.upenn.edu/~chai/papers_pdf/rig_biext_v4d.pdf`.

[19] ———, *Hecke orbits*, 2020. book project in preparation.

[20] ———, *Orbital rigidity for biextensions* (2022). Available at `https://www2.math.upenn.edu/~chai/papers_pdf/HO-Ch10-May2022.pdf`.

[21] Richard Crew, *F-isocrystals and their monodromy groups*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 4, 429–464. MR1186910

[22] Marco D'Addezio, *The monodromy groups of lisse sheaves and overconvergent F-isocrystals*, Selecta Math. (N.S.) **26** (2020), no. 3.

[23] Marco D'Addezio, *Parabolicity conjecture of F-isocrystals*, Ann. of Math. (2), to appear (2023), available at `2012.12879`.

[24] Aise Johan de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Inst. Hautes Études Sci. Publ. Math. **82** (1995), 5–96 (1996). MR1383213

[25] P. Deligne and J. S. Milne, *Tannakian categories*, Hodge cycles, motives, and shimura varieties, 1982, pp. 101–228.

[26] Vladimir Drinfeld, *A stacky approach to crystals*, arXiv e-prints (October 2018), arXiv:1810.11853, available at `1810.11853`.
[27] Vladimir Drinfeld and Kiran Kedlaya, *Slopes of indecomposable F-isocrystals*, Pure Appl. Math. Quart. **13** (2017), 131–192.
[28] Marco D'Addezio and Hélène Esnault, *On the Universal Extensions in Tannakian Categories*, International Mathematics Research Notices (202105), available at `https://academic.oup.com/imrn/advance-article-pdf/doi/10.1093/imrn/rnab107/38377494/rnab107.pdf`.
[29] Laurent Fargues and Peter Scholze, *Geometrization of the local Langlands correspondence*, arXiv e-prints (February 2021), available at `2102.13459`.
[30] Yin Nam Dalton Fung, *Hypersymmetric points on shimura varieties of hodge type*, Ph.D. thesis, Boston College (2022). available at `https://dlib.bc.edu/islandora/object/bc-ir:109363`.
[31] Wushi Goldring and Jean-Stefan Koskivirta, *Strata Hasse invariants, Hecke algebras and Galois representations*, Invent. Math. **217** (2019), no. 3, 887–984. MR3989256
[32] Ulrich Görtz, Xuhua He, and Sian Nie, *Fully Hodge–Newton Decomposable Shimura Varieties*, Peking Math. J. **2** (2019), no. 2, 99–154. MR4060001
[33] Paul Hamacher, *The product structure of Newton strata in the good reduction of Shimura varieties of Hodge type*, J. Algebraic Geom. **28** (2019), no. 4, 721–749. MR3994311
[34] Paul Hamacher and Wansu Kim, *l-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients*, Math. Ann. **375** (2019), no. 3-4, 973–1044. MR4023369
[35] Xuhua He, Rong Zhou, and Yihang Zhu, *Component stabilizers of affine Deligne-Lusztig varieties in the affine Grassmannian*, 2020.
[36] S. Mohammad Hadi Hedayatzadeh, *Exterior powers of π-divisible modules over fields*, J. Number Theory **138** (2014), 119–174. MR3168925
[37] Gerhard P. Hochschild, *Basic theory of algebraic groups and Lie algebras*, Graduate Texts in Mathematics, vol. 75, Springer-Verlag, New York-Berlin, 1981. MR620024
[38] Harald Hofstätter, *Denominators of coefficients of the Baker-Campbell-Hausdorff series*, arXiv e-prints (October 2020), available at `2010.03440`.
[39] Pol van Hoften, *On the ordinary Hecke orbit conjecture*, arXiv e-prints (December 2021), 2112.12422.
[40] Pol van Hoften and Luciena Xiao Xiao, *Monodromy and Irreducibility of Igusa Varieties*, arXiv e-prints (February 2021), 2102.09870.
[41] Pol van Hoften and Rong Zhou, *Mod p points on Shimura varieties of parahoric level*, arXiv e-prints (October 2020), available at `2010.10496`.
[42] Serin Hong, *On the Hodge-Newton filtration for p-divisible groups of Hodge type*, Math. Z. **291** (2019), no. 1-2, 473–497. MR3936079
[43] Luc Illusie, *Complexe cotangent et déformations. II*, Lecture Notes in Mathematics, Vol. 283, Springer-Verlag, Berlin-New York, 1972. MR0491681
[44] ______, *Déformations de groupes de Barsotti-Tate (d'après A. Grothendieck)*, 1985, pp. 151–198. Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84). MR801922
[45] Guy Kapon and Lior Hadassy, *Characterising Semi-Simple Lie Algebras by Their Borel Nilpotent Radical*, arXiv e-prints (November 2021), available at `2111.03323`.
[46] Nicholas M. Katz, *Slope filtration of F-crystals*, Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. I, 1979, pp. 113–163. MR563463
[47] Wansu Kim, *On central leaves of Hodge-type Shimura varieties with parahoric level structure*, Math. Z. **291** (2019), no. 1-2, 329–363. MR3936073
[48] ______, *On central leaves of Hodge-type Shimura varieties with parahoric level structure*, Math. Z. **291** (2019), no. 1-2, 329–363.
[49] Wansu Kim and Keerthi Madapusi Pera, *2-adic integral canonical models*, Forum Math. Sigma **4** (2016), Paper No. e28, 34. MR3569319
[50] M. Kisin and G. Pappas, *Integral models of Shimura varieties with parahoric level structure*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 121–218. MR3905466
[51] Mark Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), no. 4, 967–1012. MR2669706
[52] ______, mod p *points on Shimura varieties of abelian type*, J. Amer. Math. Soc. **30** (2017), no. 3, 819–914. MR3630089
[53] Mark Kisin, Yeuk Hay Joshua Lam, Ananth N. Shankar, and Padmavathi Srinivasan, *Finiteness of reductions of Hecke orbits*, arXiv e-prints (September 2021), available at `2109.05147`.
[54] Mark Kisin, Keerthi Madapusi Pera, and Sug Woo Shin, *Honda–Tate theory for Shimura varieties*, Duke Mathematical Journal **171** (2022), no. 7, 1559 –1614.

[55] Arno Kret and Sug Woo Shin, *H^0 of Igusa varieties via automorphic forms*, arXiv e-prints (February 2021), available at 2102.10690.
[56] Elena Mantovan, *On the cohomology of certain PEL-type Shimura varieties*, Duke Math. J. **129** (2005), no. 3, 573–610. MR2169874
[57] Akhil Mathew, *Faithfully flat descent of almost perfect complexes in rigid geometry*, arXiv e-prints (December 2019), available at 1912.10968.
[58] Hideyuki Matsumura, *Commutative algebra*, W. A. Benjamin, Inc., New York, 1970. MR0266911
[59] Davesh Maulik, Ananth N. Shankar, and Yunqing Tang, *Picard ranks of K3 surfaces over function fields and the Hecke orbit conjecture*, Invent. Math. **228** (2022), no. 3, 1075–1143. MR4419630
[60] J. S. Milne, *Introduction to Shimura varieties*, Harmonic analysis, the trace formula, and Shimura varieties, 2005, pp. 265–378. MR2192012
[61] Ben Moonen, *Linearity properties of Shimura varieties. II*, Compositio Math. **114** (1998), no. 1, 3–35. MR1648527
[62] ———, *Serre-Tate theory for moduli spaces of PEL type*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 2, 223–269. MR2061781
[63] Yasuhiro Oki, *On the connected components of Shimura varieties for CM unitary groups in odd variables*, arXiv e-prints (April 2021), available at 2104.13086.
[64] Frans Oort, *Foliations in moduli spaces of abelian varieties*, J. Amer. Math. Soc. **17** (2004), no. 2, 267–296. MR2051612
[65] ———, *Appendix 1: Some questions in algebraic geometry*, Open problems in arithmetic algebraic geometry, [2019] ©2019, pp. 263–283. Reprint of 1995 original. MR3971187
[66] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen. MR1278263
[67] M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. **103** (1996), no. 2, 153–181. MR1411570
[68] Peter Scholze and Jared Weinstein, *Moduli of p-divisible groups*, Camb. J. Math. **1** (2013), no. 2, 145–237. MR3272049
[69] Ananth N. Shankar, *The Hecke-orbit conjecture for "modèles étranges"*, 2016. preprint.
[70] Ananth N. Shankar and Rong Zhou, *Serre-Tate theory for Shimura varieties of Hodge type*, Math. Z. **297** (2021), no. 3-4, 1249–1271. MR4229601
[71] Xu Shen, Chia-Fu Yu, and Chao Zhang, *EKOR strata for Shimura varieties with parahoric level structure*, Duke Math. J. **170** (2021), no. 14, 3111–3236. MR4319228
[72] Xu Shen and Chao Zhang, *Stratifications in good reductions of Shimura varieties of abelian type*, arXiv e-prints (July 2017), available at 1707.00439.
[73] The Stacks project authors, *The stacks project*, 2022. https://stacks.math.columbia.edu.
[74] Torsten Wedhorn, *On Tannakian duality over valuation rings*, J. Algebra **282** (2004), no. 2, 575–609. MR2101076
[75] Luciena Xiao Xiao, *On The Hecke Orbit Conjecture for PEL Type Shimura Varieties*, arXiv e-prints (June 2020), available at 2006.06859.
[76] Yujie Xu, *Normalization in integral models of Shimura varieties of Hodge type*, arXiv e-prints (July 2020), available at 2007.01275.
[77] Chia-Fu Yu, Ching-Li Chai, and Frans Oort, *Stratifying Lie strata of Hilbert modular varieties*, Taiwanese J. Math. **24** (2020), no. 6, 1307–1352. MR4176876
[78] Rong Zhou, *Motivic cohomology of quaternionic Shimura varieties and level raising*, Ann. Sci. École Norm. Sup (2022), To appear, available at 1901.01954.

Institut de Mathématiques de Jussieu-Paris Rive Gauche, SU - 4 place Jussieu, Case 247, 75005 Paris

Email address: daddezio@imj-prg.fr

Stanford mathematics department, 450 Jane Stanford way, Building 380, Stanford, CA 94305, USA

Email address: pol.van.hoften@stanford.edu